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Service-oriented decisions on inventory levels in the case of incomplete demand information

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Abstract New products without historical demand information or slow-moving items with little such information cause difficulties in defining inventory management policies facing demand uncertainty. The classical approach using the Normal distribution for describing the random demand during lead time might lead to a degraded level of customer service. But the choice for other types of distributions is also no option, so it is realistic that the full functional form of the distribution is unknown, but the decision-maker has some but not incomplete information on the demand distribution during lead time. As the distribution is only partially specified, several distributions satisfy the known information. Customer service measures therefore also take values in an interval between a lower and an upper bound. In this paper, upper and lower bounds are determined for two performance measures: the number of stock-out units and the stock-out probability per replenishment cycle, given incomplete information about the demand distribution, that is only the first two moments and the range, are known. Based on these results, the optimal inventory level given the desired maximum number of stock-out units or the desired maximum stock-out probability is calculated for the case where only the first two moments are known. The results of our approach are compared to the more traditional approach where a Normal distribution of demand during lead time is assumed. Comparisons with the Gamma, Uniform and symmetric triangular distribution are made. Furthermore, the robustness of our bounds to uncertainty in the parameters is tested.

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1 Introduction

Uncertainty in inventory systems may be due to suppliers or customers. On the suppliers' side, uncertainty (such as lead time, yield and quality) asks for corrective action. Decisions on lot sizing with uncertain yield are important especially in production/manufacturing systems (e.g. [24]). Uncertainty, which is attributable to customers, relates especially to demand. If insufficient inventory is held, a stock-out may occur leading to shortage costs or customer service degradation. As shortage costs are usually high in relation to holding costs, companies hold additional inventory, above their forecasted needs, to add a margin of safety. Some decision models combine both the uncertainty of yield and demand (e.g. [12]).

Determination of an inventory replenishment policy, of the quantities to order, of the review period is typical decisions to be taken by logistics managers. Decisions are taken making use of optimization models taking a performance characteristic into consideration, which might be cost-oriented or service-oriented. A cost-oriented model translates non-satisfied demand into a shortage or backorder cost. A comprehensive review of the earlier backorder-cost inventory models can be found in Federgruen [5]. But estimating the costs associated with backordering and loss of customer goodwill is difficult. Hence, service-constrained inventory problems received more attention. A review appears in Diks et al. [4], and for distribution systems, Ozer and Xiong [16] may be consulted. Performance characteristics of the service-oriented type may be expressed relatively as the proportion of customer demand met from inventory, or may be expressed absolutely in terms of number of stock-out units, which is a direct indication for lost sales. Also the probability of having a stock-out in a replenishment period is often used as a performance characteristic.

The probability distribution of demand is an important input in inventory management. In literature, it is extensively studied. However, in practice, a logistics manager often faces incomplete information on the distribution of demand during lead time. This research deals with the case where the demand distribution during lead time is not completely specified. This situation is realistic either with products that have been introduced recently to the market or with slow-moving products, for example service parts. In general, with recently introduced products or slow movers, no sufficient historical data are available to decide on the functional form of the demand distribution function. This fact does not imply that no information exists about the demand distribution as incomplete (or partial) information might exist like the range of the demand, its expected value and/or variance.

This paper concentrates on two service performance measures: the expected number of stock-out units and the stock-out probability (in a replenishment period). In the following sections, lower and upper bounds are obtained for the performance measures under study, given various levels of information about the demand distribution. From a production or trading company's point of view, a decision might be formulated to answer the following question: given an expected number of stock-out units or a stock-out probability the company wants to face, what should be the safety inventory at least or at most?

The organization of the paper is as follows: in Sect. 2, an overview of related literature is provided; Sect. 3 describes the method used to calculate the bounds; in Sect. 4, the number of stock-out units is discussed; in Sect. 5, the stock-out probability is dealt with; in Sect. 6, the robustness of the results is tested and in Sect. 7, conclusions are drawn from the study.

2 Probability distribution of demand during lead time

In reorder point models for inventory management, the probability distribution of demand is a vital characteristic. A decision-maker using an inventory model including uncertainty on demand must select a probability distribution as input to the model. But in reality, it is often difficult to completely characterize the distribution, especially in the case of little historical data. Most textbooks assume that the demand for an item is formed by a large number of smaller demands from individual customers. As a result, many authors assume that the resulting demand size in a certain period of time is a continuous random variable and follows a Normal distribution. In practice, it has been shown that, for fast-moving items, a Normal distribution is appropriate.

The use of the Normal distribution for a demand size distribution can be questioned because (1) the distribution is defined both on the positive and negative axes; and (2) the distribution is symmetrical. While the Normal distribution may be approximately correct in many cases, it is conceptually not. It cannot be used in computer simulation as negative demand may be generated at random. When of relevance, one rather should look for a probability distribution, which is defined only for non-negative values and allows for skewness.

But if the Normal distribution is not considered appropriate, the question arises which alternative distribution to select as they may lead to other values of the decision variables, like the order quantity. In the literature on inventory control, many times reference is made to the Gamma distribution. It is defined only on non-negative values and, according to the parameters of its distribution, ranges in shape from a monotonic decreasing function, through unimodal distributions skewed to the right, to the Normal distribution. The Gamma distribution is attractive because of the ease it can deal with fixed lead times and how the situations can be extended to probabilistic lead times. For items with low demand, Laplace or Poisson distributions are proposed [20]. The Poisson distribution has been found to provide a reasonable fit when the demand is very low (only a few pieces per year).

But when demand frequency is not too high, an alternative approach is offered by the use of separate distributions for the demand occurrence and for the demand size. Models have been developed using the Poisson distribution for the demand occurrence. When the demand size is described by an arbitrary probability distribution and the demand occurrence process is described as a Poisson process, the total demand during a finite time period can be described by a compound Poisson distribution. A two-echelon spare parts inventory systems has been studied by Shanker [19] where the warehouse acts as a centralized repair facility and the depot faces a compound Poisson demand.

A case study by Vereecke and Verstraeten [22] shows that demand variance often is a multiple of the average demand, showing that the Poisson distribution is not a good approximation of the demand size. They propose a construction called the 'Package Poisson', where the average demand is expressed in numbers of packages of fixed size. The size of the package is defined by using empirical data on both the average and variance of the demand in terms of units.

This paper deals with the case where the demand distribution during lead time is not completely specified. This situation is realistic either with products that have been introduced recently to the market or with slow-moving products. In both cases, insufficient data are available to decide on the functional form of the demand distribution function. In such cases, the assumption of a Normal distribution shape of the demand distribution might be violated. Even in a well-defined class of distributions, like the case of compound Poisson distributions, given some moments of the demand distribution, the shape might range from close to Normal, over very skewed distributions even to bimodal distributions [18]. Bartezzaghi et al. [1] show a significant impact of the demand shape on required inventory levels to achieve a predefined service level. The coefficient of variation is a constant in their experiments. The analysis shows that the demand shape is a primary factor in the determination of the inventories and that the impact of different demand shapes on inventories is comparable to the effect of doubling the coefficient of variation. This means that even within this limited class of distributions, like for example the compound Poisson distribution, which is of application to slow-moving items, attention should be paid to the parameters influencing the shape. The experiments by Bartezzaghi et al. [1] are in conflict with earlier, more limited studies by Naddor [15] and by Fortuin [7] who observed that inventory decisions are relatively insensitive to the choice of distribution, when the mean and variance are specified. This fact keeps the discussion alive.

The aspect of incomplete information has been addressed several times in literature, for example in the single-period (newsvendor) inventory problem. It has a wide variety of applications in industry, typically for products with short lives, like fashion goods, consumer electronics, which are rapidly evolving as cellular phones, or vaccines against a single season influenza. Walker [23] developed a decision support tool for the single-period inventory problem. While the newsvendor problem is easily solved in case of full demand information, mostly in practice, this information is hard to get in a such real-world applications. In case only mean and variance of demand are known, Gallego and Moon [8] propose an approach that maximizes the worst case profit. Vairaktarakis [21] uses another performance criterion, that is trying to minimize the maximum regret in the case only a lower and upper bound on the demand are known. Perakis and Roels [17] use a minimax regret approach when only partial demand information is known, like mean, standard deviation or unimodality.

A number of papers deal with the estimation of demand uncertainty when historical demand data are not available. Possible approaches are using historical data sets of other products and advance purchase orders for new products [13, 14] or using expert opinions on the products to be offered [6, 9].

In this paper, we address the case where insufficient data are available to decide on the functional form of the demand distribution function. This fact does not imply that no information exists about the demand distribution as incomplete (or partial) information might exist like the range of the demand, its expected value and/or variance. Under this condition of incomplete information on demand, the optimal inventory level is determined given a desired performance level.

3 Methodology to support inventory decision-making in case of uncertain demand

In the following sections, a methodology is developed to support the decision-maker in finding the best values of decision variables in the case of uncertainty in demand. It should be stressed that this approach does not make use of a cost function in which shortage costs or loss of customer confidence are included but only performance measures are included. The sections concentrate on two service performance measures: the expected number of stock-out units and the stock-out probability. Both measures can be expressed as the expected value of a function, where the expected value is generated by the probability distribution of the demand during lead time. In case full knowledge exists about this distribution, both performance measures can be computed as a single value. In case only incomplete information exists about the distribution, the performance measures take different values for distributions with the same characteristics, so no single value can be computed, but rather a range of values. This fact raises the question whether this range is finite and, if yes, what are the lower and upper bounds of this range. This information is of great use to the decision-maker to make his final decision on decision variables like the re-order point. Therefore, a methodology is used in which, for both performance measures, upper and lower bounds can be computed as support information for the decision-maker. Upper bounds correspond to a pessimistic viewpoint of the decision-maker; lower bounds correspond to an optimistic viewpoint of the decision-maker. Depending on the degree of optimism of the decision-maker, any value between the upper and the lower bound on the inventory level can be used to determine decision variables like the re-order point. By using the upper bounds in his decisions, the decision-maker protects himself completely as no demand distribution exists to a higher expected number of stock-out unit or a higher stock-out probability.

Let X be a random variable indicating the demand during lead time and d be the inventory level at the start of a lead time (henceforth referred to as inventory level). Further, let W be the number of stock-out units. The relationship between W and X is given by:

$$W = \begin{cases} 0 & \text{if } X \le d; \\ X - d & \text{if } X > d. \end{cases} \tag{1}$$

The expected number of stock-out units during a lead time is defined as $E((X-d)_+) = \int_a^b (x-d)_+ \, \mathrm{d}F(x) = \int_d^b (x-d) \, \mathrm{d}F(x)$, where E is the expected value operator, $F(x)$ is the cumulative probability distribution with support on [a,b] (with $a > 0$, $b \ge a$, $a \le d \le b$ and $a, b < +\infty$). When a company holds d units of a specific product in inventory starting a period between order and delivery, any demand X less than d is satisfied, while any demand X greater than d results in a shortage of $X - d$ units. A lesser number of stock-out units results in a better service to the customer.

From a mathematical viewpoint, the expected number of stock-out units has a counterpart in insurance mathematics as the stop-loss premium. Using the same symbols, it can be stated that a stop-loss premium limits the risk X of an insurance company to a certain amount d [11]. In that field, results have been obtained for deriving lower and upper bounds on the stop-loss premium where the risk is allowed to vary under some constraints such as given known range, first and second order moments, unimodality etc. [3, 10].

If U is an indicator for the stock-out probability, the relationship between U and X is given by:

$$U = \begin{cases} 0 & \text{if } X \le d; \\ 1 & \text{if } X > d. \end{cases} \tag{2}$$

The stock-out probability is defined as

$$P(X \ge d) = E[1_{[d,b]}(X)] = \int_a^b 1_{[d,b]}(x) \, \mathrm{d}F(x) = \int_d^b \mathrm{d}F(x)$$

Bounds on the stock-out probability during a replenishment period in inventory management can be simply translated into bounds on tail probabilities. Upper and lower bounds on tail probabilities under varying levels of information on the demand distribution like mode, mean and/or variance have been deduced by De Schepper and Heijnen [2].

Before moving towards the remainder of this paper, it should be stated that the bounds and their use in applications can be translated from any distribution defined on $[a, b]$ (with $a > 0$, $b \ge a$ and $a, b < +\infty$) into the bounds with a distribution defined on $[0, b_0]$. Furthermore, the first two moments of the distribution are assumed to be known. Let $\mu_1 = E(X)$ and $\mu_2 = E(X^2)$. If $a \ne 0$ and a, b, μ_1 and μ_2 are known, the parameters for the distribution defined on $[0, b_0]$ can be calculated using the following formulas:

$$a_0 = 0, \tag{3}$$

$$b_0 = b - a, \tag{4}$$

$$\mu_{1,0} = \mu_1 - a, \tag{5}$$

$$\mu_{2,0} = \mu_2 - 2a\mu_1 + a^2. \tag{6}$$

The following paragraphs, without loss of generalization, make use of distributions defined on $[0, b_0]$. Furthermore, b will be used instead of b_0 since there is no risk of confusion.

The demand X is a positive random variable with an upper bound b. From a mathematical point of view, when calculating bounds on both performance measures, the problem is to find

$$\sup_{F \in \Phi} \int_0^b f(x) \, \mathrm{d}F(x) \tag{7}$$

and

$$\inf_{F \in \Phi} \int_0^b f(x) \, \mathrm{d}F(x) \tag{8}$$

where Φ is the class of all distribution functions with range $[0,b]$ and moments μ_1 and μ_2 and where $f(x) = (x - d)_+$ in the case of the expected number of stock-out units and $f(x) = 1_{[0,b]}(x)$ in the case of the stock-out probability.

For any polynomial $P(x)$ of degree 2 or less, the integral $\int_0^b P(x) \, \mathrm{d}F(x)$ only depends on μ_1 and μ_2, so it takes the same value for all distributions in Φ. Polynomials P are looked for such that

- $P \ge f$ on $[0,b]$ (in case of upper bound) or $P \le f$ on $[0,b]$ (in case of lower bound)
- there is some distribution G in Φ for which equality holds:

$$\int_0^b P(x) \, \mathrm{d}G(x) = \int_0^b f(x) \, \mathrm{d}G(x) \tag{9}$$

As distribution G, a two- or three-point distribution is used. For such distributions, the equality mentioned above is attained when $P(x)$ and $f(x)$ are equal in the mass points of G. The best upper and lower bounds on this term with given moment μ_1 and μ_2 are derived. To apply this method, the formula for a unique parabola $g(x)$ taking values $f(u)$ and $f(v)$ in u and v with derivative $f'(v)$ in v (u and v any points in $[0,b]$) is needed:

$$\begin{aligned} g(x) = \frac{1}{(v-u)^2} &[f(v)(v-u)(x-u) + f(u)(u-v)(x-v) \\ &+ [f'(v)(v-u) - f(v) + f(u)](x-u)(x-v)]. \end{aligned} \tag{10}$$

The deductions for the expected number of stock-out units [3, 10] and the expected stock-out probability [2] can be found in Appendix.

4 Expected number of stock-out units

Using the method described in the previous section, upper and lower bounds on the expected number of stock-out units can be obtained, given the inventory level and incomplete information about the demand distribution. However, for a production or trading company, the inverse problem is more important. They are interested to know, given an expected number of stock-out units the company wants to face, what the inventory level should be at least or at most.

First, the case of a known (finite) range, expected value and variance is discussed: upper and lower bounds on the number of stock-out units are determined, and the optimal inventory level is calculated given the desired maximum number of stock-out units. Afterwards, a numerical example is used to illustrate the use of the bounds. In the last subsection, the results are compared with results for other distributions: the Normal distribution, the Gamma distribution, the Uniform distribution and the symmetric triangular distribution.

4.1 The case of a known (finite) range, expected value and variance

Table 1 shows the results for the upper bounds. The domain of the parameters is

$$0 \leq \mu_1 \leq b \tag{11}$$

and

$$\mu_1^2 \leq \mu_2 \leq \mu_1 b. \tag{12}$$

Further, let μ_1 and μ_2 be chosen that inequalities (11) and (12) hold, then let $r' = (\mu_2 - \mu_1 r)/(\mu_1 - r)$ for every $r \in [0, b]$ and $r \neq \mu_1$.

Table 2 shows the results for the lower bounds. The domain of the parameters is the same as in Table 1.

Table 1 Upper bounds on the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Upper bound
$d \leq \frac{0'}{2}$	$\frac{\mu_1}{\mu_2}(\mu_2 - \mu_1 d)$
$\frac{0'}{2} \leq d \leq \frac{b+b'}{2}$	$\frac{\mu_1 - d + \sqrt{(\mu_2-\mu_1^2)+(d-\mu_1)^2}}{2}$
$d \geq \frac{b+b'}{2}$	$\frac{(\mu_2-\mu_1^2)(b-d)}{(\mu_2-\mu_1^2)+(b-\mu_1)^2}$

Table 2 Lower bounds on the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Lower bound
$0 \leq d \leq b'$	$\mu_1 - d$
$b' < d < 0'$	$\frac{\mu_2 - \mu_1 d}{b}$
$0' \leq d \leq b$	0

Table 3 Optimal inventory level using the upper bounds of the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Inventory level
$E(W) \leq \frac{\mu_2-\mu_1^2}{2(b-\mu_1)}$	$b - \frac{E(W)[(\mu_2-\mu_1^2)+(b-\mu_1)^2]}{\mu_2-\mu_1^2}$
$\frac{\mu_2-\mu_1^2}{2(b-\mu_1)} \leq E(W) \leq \frac{\mu_1}{2}$	$\frac{(\mu_2-\mu_1^2)-4E(W)^2+4E(W)\mu_1}{4E(W)}$
$E(W) \geq \frac{\mu_1}{2}$	$\frac{(\mu_1-E(W))\mu_2}{\mu_1^2}$

Table 4 Optimal inventory level using the lower bounds of the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Inventory level
$E(W) \leq \frac{\mu_2-\mu_1^2}{b-\mu_1}$	$\frac{\mu_2 - bE(W)}{\mu_1}$
$E(W) \geq \frac{\mu_2-\mu_1^2}{b-\mu_1}$	$\mu_1 - E(W)$

From Tables 1 and 2, upper bounds on d, which correspond to a pessimistic viewpoint, and lower bounds on d, which correspond to an optimistic viewpoint, can be derived. In Table 3, the optimal inventory level is expressed in terms of the desired number of stock-out units, using the upper bounds on the expected number of stock-out units. By choosing the inventory levels in Table 3, the decision-maker protects himself completely as no demand distribution exists leading to a higher expected number of stock-out units. In Table 4, the lower bounds on the expected number of stock-out units are used to calculate the optimal inventory level in terms of the desired number of stock-out units. By choosing the levels in Table 4, the decision-maker nearly always will be faced with an expected number of stock-out units higher than the target.

Depending on the degree of optimism of the company, any value between the upper and lower bound on the inventory level can be used to determine the inventory level that needs to be held at the beginning of a lead time period.

4.2 Numerical example

In this section, a numerical example demonstrates the use of bounds on performance measures. In this example, the following information on demand during lead time is

Table 5 Numerical example of upper bounds on the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Upper bound
$0 \leq d \leq 14.5$	$25 - \frac{25}{29}d$
$14.5 \leq d \leq 35.5$	$12.5 - \frac{1}{2}d + \frac{1}{2}\sqrt{100 + (d-25)^2}$
$35.5 \leq d \leq 50$	$\frac{200-4d}{29}$

Table 6 Numerical example of lower bounds on the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Lower bound
$0 \leq d \leq 21$	$25 - d$
$21 \leq d \leq 29$	$14.5 - \frac{1}{2}d$
$29 \leq d \leq 50$	0

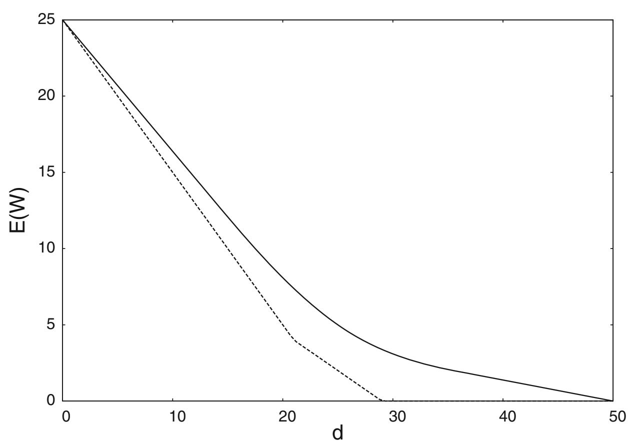

Fig. 1 Upper and lower bounds on expected number of stock-out units given the inventory level d

known: the mean $\mu_1 = 25$, the second moment $\mu_2 = 725$ and the range of demand is $[0,b]$ with $b = 50$.

The upper and lower bounds on the number of stock-out units are presented in Tables 5 and 6. Figure 1 shows the upper and lower bounds on the expected number of stock-out units for a given inventory level.

Tables 7 and 8 show the optimal inventory level, given the desired level of maximum number of stock-out units. Figure 2 shows the upper and lower bounds on the inventory level for a given expected number of stock-out units.

These results are illustrated in a specific case. If, for example, the company wants to face a maximum of 5 stock-out units in a period, the upper bound on the inventory level equals 25 and the lower bound on the inventory level equals 20. This means that if the company is very risk averse, an inventory level of 25 units is held, and if the company is more risk seeking, an inventory level of 20 units is held.

Table 7 Numerical example of the optimal inventory level using the upper bounds on the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Inventory level
$E(W) \leq 2$	$50 - \frac{29}{4}E(W)$
$2 \leq E(W) \leq 12.5$	$\frac{25-E(W)^2+25E(W)}{E(W)}$
$E(W) \geq 12.5$	$\frac{725-29E(W)}{25}$

Table 8 Numerical example of the optimal inventory level using the lower bounds on the expected number of stock-out units when μ_1 and μ_2 are known

Conditions	Inventory level
$E(W) \leq 4$	$29 - 2E(W)$
$E(W) \geq 4$	$25 - E(W)$

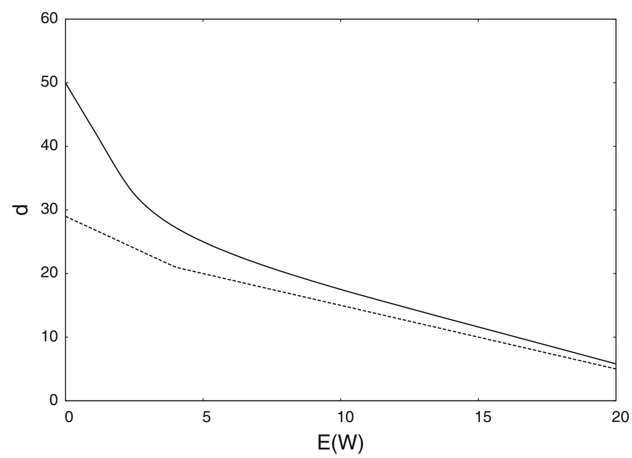

Fig. 2 Optimal inventory level using upper and lower bounds on the expected number of stock-out units

4.3 Comparison with results for other distributions

In this section, the results above are compared to results obtained by using the Normal distribution, the Gamma distribution, the Uniform distribution and the symmetric triangular distribution.

In literature, for fast-moving items, the safety stock is mostly determined using the Normal distribution for describing demand during lead time. Therefore, the results are first compared to results for the Normal distribution. Determining bounds on the inventory level using the number of stock-out units corresponds to using the service level approach for the Normal distribution. For the same mean and variance, the inventory level is calculated using the service level approach, and the bounds on the inventory level are calculated using the approach above. Calculations are made for the maximum number of stock-out units in a

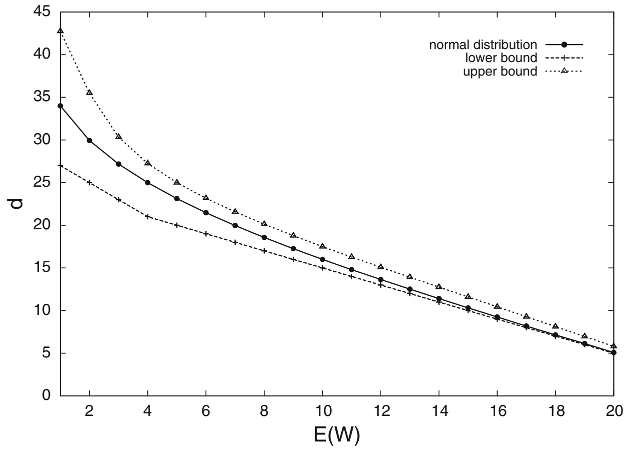

Fig. 3 Comparison of service level approach with lower and upper bounds

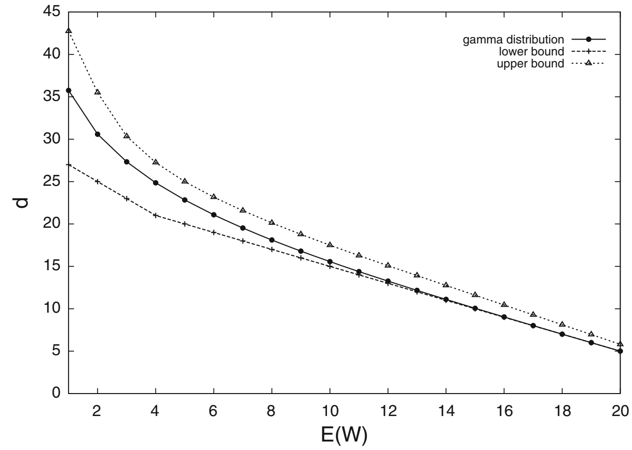

Fig. 5 Comparison of the Gamma distribution with lower and upper bounds

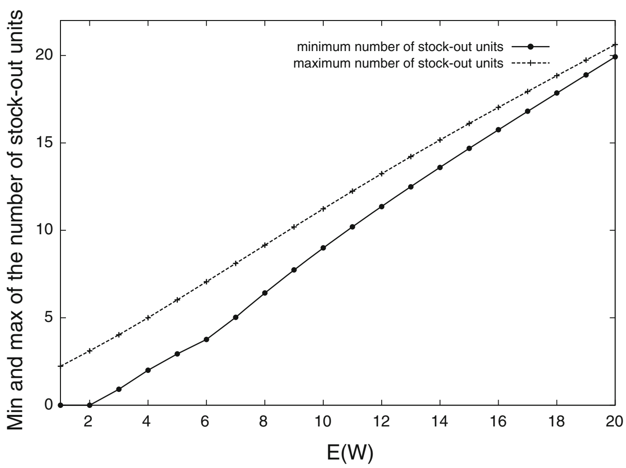

Fig. 4 Range of possible values for the number of stock-out units when using the service level approach to calculate the inventory level

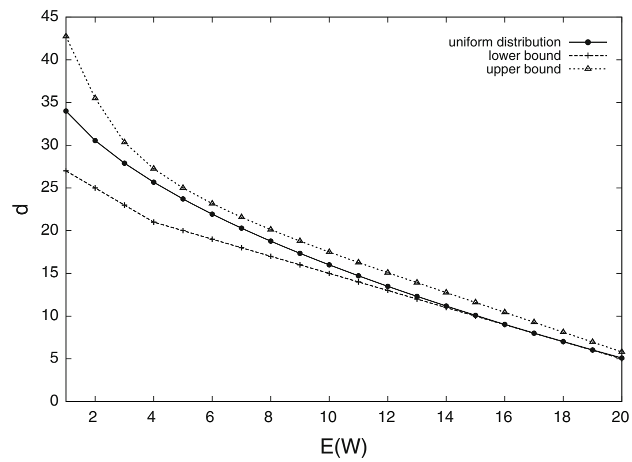

Fig. 6 Comparison of the Uniform distribution with lower and upper bounds

period ranging from 1 to 20. The results are visualized in Fig. 3. However, when the demand distribution during lead time is not completely known, the use of the Normal distribution may result in more stock-out units than assumed. In Fig. 4, results are shown for the minimum and maximum number of stock-out units when the inventory level is determined by the service level approach (the advice from Fig. 3). For example, if a maximum number of 3 stock-out units is targeted, the service level approach leads to an inventory level of 27 (see Fig. 3). However, when the demand distribution is not completely known, the actual number of stock-out units for an inventory level of 27 varies between 0.91 and 4.03.

The same calculations are done for the Gamma distribution, the Uniform distribution and the symmetric triangular distribution. The parameters of all three distributions are chosen in such a way that the mean and variance correspond to the mean and variance used in the calculations of the example in the previous section. The parameters for the Gamma distribution are $\alpha = 6.25$ and $\beta = 4$. The range of the Uniform distribution is [7.68,42.32]. The symmetric triangular distribution has a lower limit $a = 0.5$, an upper limit $b = 49.5$ and a mode $c = 25$. Calculations are made for the maximum number of stock-out units in a period ranging from 1 to 20. The results are visualized in Figs. 5, 6 and 7. For all three distributions, we can conclude that using one of these distributions to determine the inventory level may lead to more stock-out units than assumed.

5 Stock-out probability

In this section, the methodology described in Sect. 3 is used to obtain upper and lower bounds on the stock-out probability in inventory management, given the inventory level and incomplete information about the demand distribution.

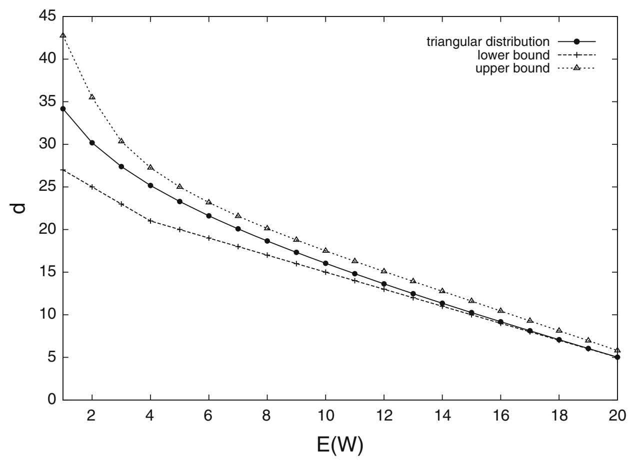

Fig. 7 Comparison of the symmetric triangular distribution with lower and upper bounds

Again, from a production or trading company's point of view, it is more interesting to know, given an expected stock-out probability the company wants to face, what the inventory level should be at least or at most.

First the case of a known (finite) range, expected value and variance is discussed. Next, the use of the bounds is illustrated by means of a numerical example. In the last subsection, the results are compared with results for the Normal distribution.

5.1 The case of a known finite range, expected value and variance

The range of the variable is [0,b] with $b > 0$. This means that the domain of the parameters is

$$0 \leq \mu_1 \leq b \tag{13}$$

and

$$\mu_1^2 \leq \mu_2 \leq \mu_1 b. \tag{14}$$

Further, let μ_1 and μ_2 be chosen that the inequalities (13) and (14) hold, then let $r' = (\mu_2 - \mu_1 r)/(\mu_1 - r)$ for every $r \in [0, b]$ and $r \neq \mu_1$.

Table 9 shows the results for the upper and lower bounds.

The results in Table 9 can be used to determine the optimal inventory level given a desired maximum stock-out probability. Table 10 presents the results for the optimal inventory level, given the desired stock-out probability, using the upper bounds on the expected stock-out probability. Table 11 shows the optimal inventory level in terms of the desired stock-out probability, using the lower bound on the expected stock-out probability.

The choice for the upper bound to calculate the required inventory level reflects a pessimistic view, while the choice for the lower bound reflects an optimistic view.

Table 9 Upper and lower bounds on the stock-out probability when μ_1 and μ_2 are known

Conditions	Upper bound	Lower bound
$0 \leq d \leq b'$	1	$\frac{(\mu_1 - d)^2}{\mu_2 - \mu_1^2 + (\mu_1 - d)^2}$
$b' < d \leq 0'$	$\frac{(b+d)\mu_1 - \mu_2}{bd}$	$\frac{\mu_2 - \mu_1 d}{b(b-d)}$
$0' < d < b$	$\frac{(\mu_2 - \mu_1^2)}{(\mu_2 - \mu_1^2) + (\mu_1 - d)^2}$	0
$d = b$	0	0

Table 10 Optimal inventory level using the upper bounds of the expected stock-out probability when μ_1 and μ_2 are known

Conditions	Inventory level
$E(U) \leq \frac{\mu_2 - \mu_1^2}{b^2 - 2b\mu_1 + \mu_2}$	b
$\frac{\mu_2 - \mu_1^2}{b^2 - 2b\mu_1 + \mu_2} < E(U) < \frac{\mu_1^2}{\mu_2}$	$\frac{\mu_1 E(U) + \sqrt{(E(U)^2 - E(U))(\mu_1^2 - \mu_2)}}{E(U)}$
$E(U) \geq \frac{\mu_1^2}{\mu_2}$	$\frac{b\mu_1 - \mu_2}{bE(U) - \mu_1}$

Table 11 Optimal inventory level using the lower bounds of the expected stock-out probability when μ_1 and μ_2 are known

Conditions	Inventory level
$E(U) \geq \frac{\mu_2 - \mu_1^2}{b^2 - 2b\mu_1 + \mu_2}$	$\frac{\mu_1(E(U) - 1) + \sqrt{\mu_1^2(E(U)-1)^2 - (E(U)-1)(\mu_2 E(U) - \mu_1^2)}}{E(U) - 1}$
$E(U) < \frac{\mu_2 - \mu_1^2}{b^2 - 2b\mu_1 + \mu_2}$	$\frac{\mu_2 - E(U)b^2}{\mu_1 - E(U)b}$

Table 12 Numerical example of upper and lower bounds on the expected stock-out probability when μ_1 and μ_2 are known

Conditions	Upper bound	Lower bound
$0 \leq d \leq 21$	1	$\frac{(25-d)^2}{100 + (25-d)^2}$
$21 \leq d \leq 29$	$\frac{21+d}{2d}$	$\frac{725 - 25d}{50(50-d)}$
$29 \leq d < 50$	$\frac{100}{100 + (25-d)^2}$	0
$d = 50$	0	0

5.2 Numerical example

The numerical example from Sect. 4.2 is used to demonstrate the use of bounds on the stock-out probability. In the numerical example, $\mu_1 = 25$, $\mu_2 = 725$ and $b = 50$. The upper and lower bounds on the stock-out probability are presented in Table 12. Figure 8 shows the upper and lower bounds on the stock-out probability as a function of the inventory level.

Tables 13 and 14 show the optimal inventory level, given the desired level of stock-out probability. Figure 9

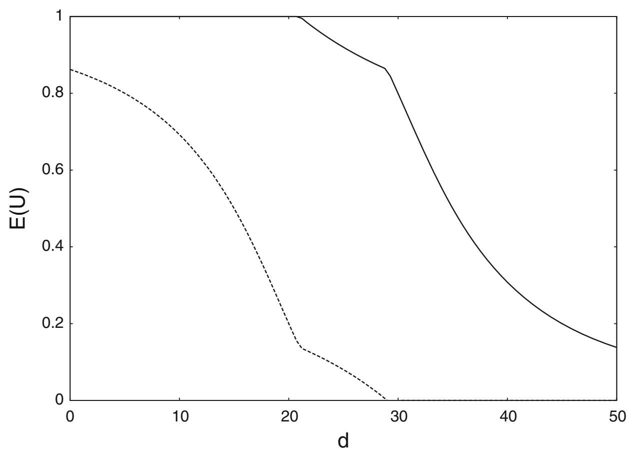

Fig. 8 Upper and lower bounds on stock-out probability given the inventory level d

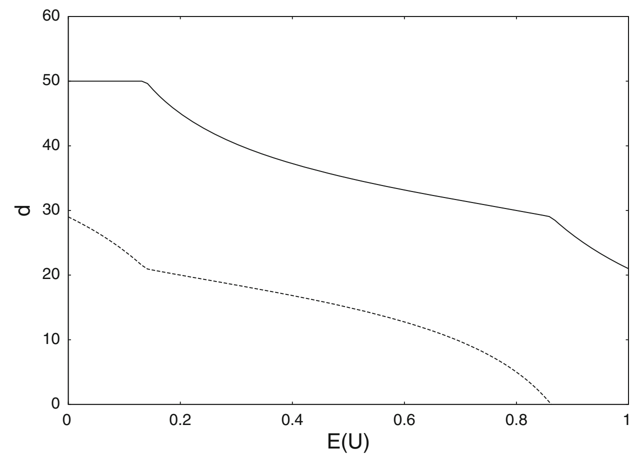

Fig. 9 Optimal inventory level using the upper and lower bounds on the stock-out probability

Table 13 Numerical example of the optimal inventory level using the upper bounds on the expected stock-out probability when μ_1 and μ_2 are known

Conditions	Inventory level
$E(U) \leq 0.138$	50
$0.138 < E(U) < 0.862$	$\frac{25E(U)+\sqrt{100(E(U)-E(U)^2)}}{E(U)}$
$E(U) \geq 0.862$	$\frac{525}{50E(U)-25}$

Table 14 Numerical example of the optimal inventory level using the lower bounds on the expected stock-out probability when μ_1 and μ_2 are known

Conditions	Inventory level
$E(U) \leq 0.138$	$\frac{29-100E(U)}{1-2E(U)}$
$E(U) \geq 0.138$	$\frac{25(E(U)-1)+\sqrt{625(E(U)-1)^2-(E(U)-1)(725E(U)-525)}}{E(U)-1}$

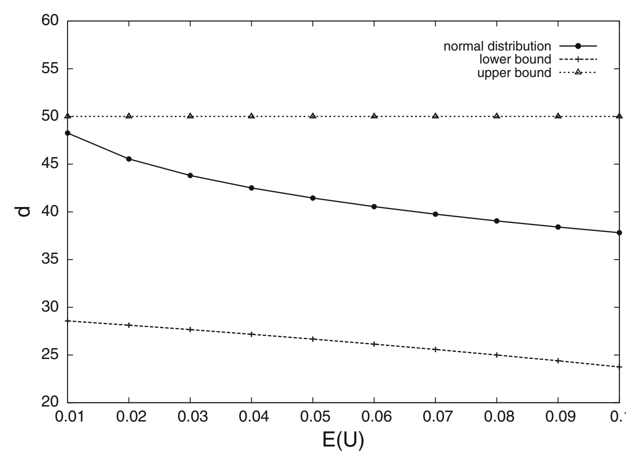

Fig. 10 Comparison of probability approach with lower and upper bounds

shows the upper and lower bounds on the optimal inventory level given a desired stock-out probability.

The use of these results is illustrated for a specific case. If the company wants to face a maximum stock-out probability of 10 % in a replenishment period, the upper bound on the inventory level equals 50 and the lower bound on the inventory level stock equals 23.75. This means that if the company is very risk averse, an inventory level of 50 units is held, and if the company is more risk seeking, an inventory level of 23.75 units is held.

5.3 Comparison with results for other distributions

Similar to Sect. 4.3, the results for the stock-out probability are compared to results obtained by using other distributions: the Normal distribution, the Gamma distribution, the Uniform distribution and the symmetric triangular distribution.

Determining bounds on the inventory level using the stock-out probability corresponds to using the probability approach for the Normal distribution. Given a mean and variance, the required inventory level is calculated using the probability approach using the Normal distribution. The upper and lower bounds on the inventory level are calculated using the approach above, with the additional assumption of finite range. Calculations are made for the stock-out probability in a period ranging from 1 to 10 %. The results are visualized in Fig. 10. However, when the demand distribution during lead time is not completely known, using the Normal distribution can result in a higher stock-out probability than assumed. In Fig. 11, results are shown for the minimum and maximum stock-out probability when the inventory level determined by the probability approach was used. For example, if a maximum of 1 % stock-out probability is presupposed, the probability approach leads to an inventory level of 48. However, when

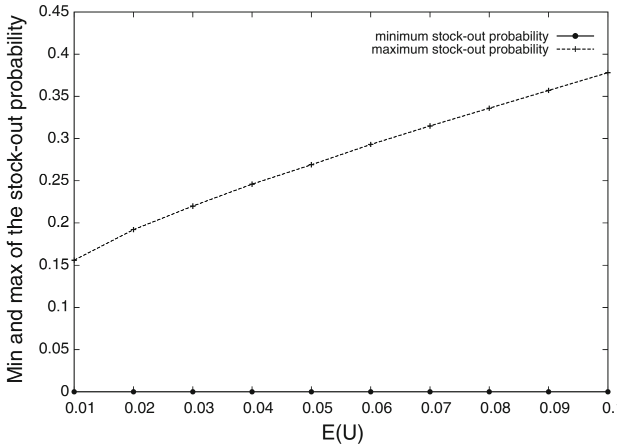

Fig. 11 Range of possible values for the stock-out probability when using the probability approach to calculate the inventory level

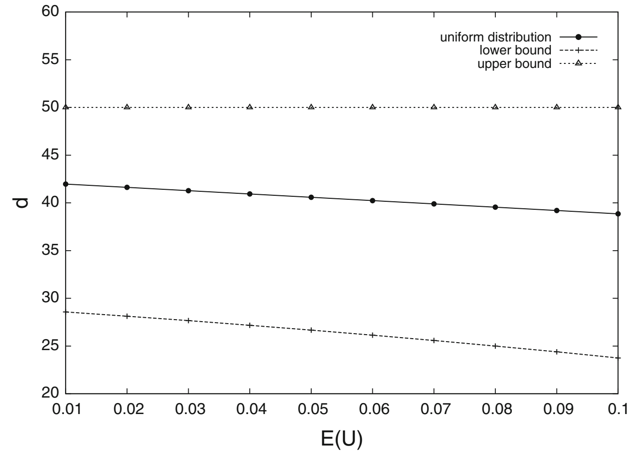

Fig. 13 Comparison of the uniform distribution with lower and upper bounds

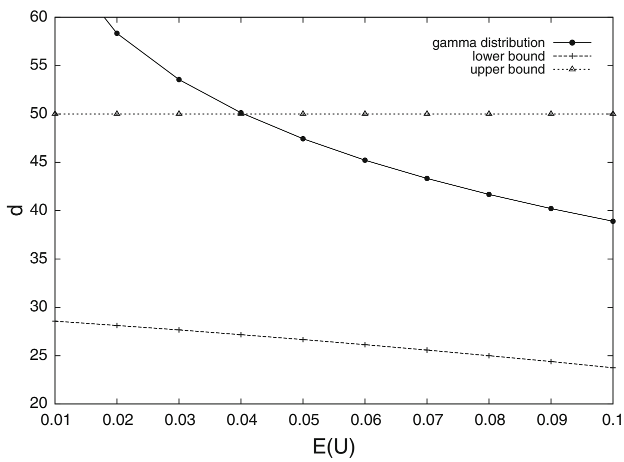

Fig. 12 Comparison of the Gamma distribution with lower and upper bounds

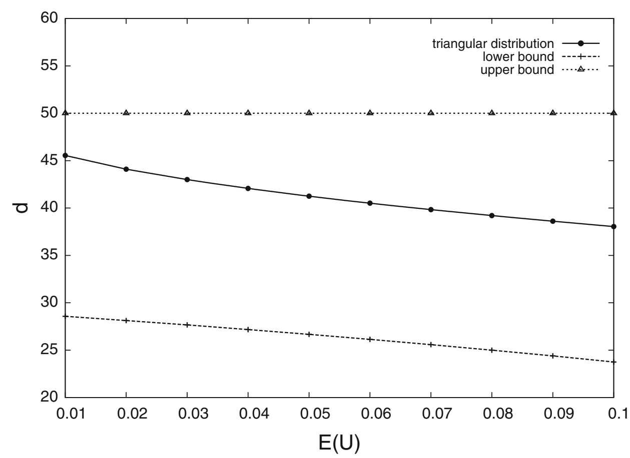

Fig. 14 Comparison of the symmetric triangular distribution with lower and upper bounds

the demand distribution is not completely known, the actual stock-out probability for an inventory level of 48 can vary between 0 and 15.6 %.

The same calculations are done for the Gamma distribution, the Uniform distribution and the symmetric triangular distribution. The parameters of all three distributions are given in Sect. 4.3. Calculations are made for the stock-out probability in a period ranging from 1 to 10 %. The results are visualized in Figs. 12, 13 and 14. For all three distributions, we can conclude that using one of these distributions to determine the inventory level may lead to a higher stock-out probability than assumed.

6 Robustness of the results

Recall that this research deals with the case where the demand distribution during lead time is not completely known: not sufficient data are available to decide on the functional form of the demand distribution function. Incomplete but not full information exists: the range of the demand, its expected value and variance. Upper and lower bounds are determined for the expected number of stock-out units and the expected stock-out probability.

In practice, however, it is improbable that, in a situation where demand information is incomplete, both the first two moments and the range are known with certainty. Errors on the estimates of both moments and on the range might be considerably high. Therefore, in this section, the effect of uncertainty in the parameter values is investigated.

By means of an experiment, we aim to investigate the influence of the estimation error on one of the incomplete demand distribution parameters (range, mean, variance) on the information offered to the decision-maker, that is an upper and lower bound on both service performance measures. The results of the experiment teach the decision-makers which parameters are more robust and which are more sensitive. In such a way, they are better informed which demand distribution parameters have an impact on the effect of changes in the mean and variance on the upper and lower bound.

An experimental design is set up to analyse the robustness of the results to changes in the parameter values (Table 15). Three factors are considered in the experimental design: the upper bound of the range b, the mean μ_1 and the coefficient of variation cv. A 2^3 full factorial design is used. The values for each factor are chosen in such a way that a wide range of combinations is examined. The upper bound of the distribution range b takes the values 50 and 100. For both values of b, cv takes the values $\sqrt{2}/2$ and $\sqrt{2}/4$. Within these four combinations, μ_1 is set equal to a low value (1/5 of the interval) and to a high value (3/5 of the interval). For all combinations, the conditions of Sect. 4 are met.

Next, for each of the experimental points, six variants are determined representing uncertainty in the parameters. In the first two variants, the mean μ_1 is increased (variant 1) or decreased (variant 2) by 10 %, while the variance changes so that the coefficient of variation remains unchanged. In the third and fourth variant, the mean μ_1 is also increased (variant 3) or decreased (variant 4) by 10 %, but the variance remains unchanged, and the coefficient of variation changes. In the last two variants, the mean μ_1 is a constant, and the standard deviation is increased (variant 5) or decreased (variant 6) by 10 %.

To check the robustness of the bounds, for each variant, new bounds are calculated and compared with the bounds of the basic scenario. Table 16 summarizes the results for the case where the expected number of stock-out units is used as a service performance measure. The results for the expected stock-out probability are shown in Table 17. In both tables, for three responses (the lower bound (ΔLB), the upper bound (ΔUB) and the length of the interval [lower bound; upper bound] ($\Delta Length$)), the changes are shown. In the columns, a '+'-symbol indicates an increase in the response value, a '−'-symbol indicates a decrease in the response values and a '='-symbol indicates that the response value remains unchanged. By using the symbols instead of numerical values, obtained through the experiments, the influence of each experimental variant is easier to interpret. Maybe it should be mentioned here that the simulations based on the variants defined are of deterministic nature. As the changes are of no stochastic origin, no significance interpretation should be given to the symbols indicating increase, decrease or no-change.

The results for the expected number of stock-out units show that for variants 1–4, the upper and lower bound increase when μ_1 is increased (variants 1 and 3), and the upper and lower bound decrease when μ_1 is decreased (variants 2 and 4). Only the changes in the length of the

Table 15 Experimental design

Experimental point	b	cv	μ_1
1	50	$\sqrt{(2)}/2$	10
2	50	$\sqrt{(2)}/2$	30
3	50	$\sqrt{(2)}/4$	10
4	50	$\sqrt{(2)}/4$	30
5	100	$\sqrt{(2)}/2$	20
6	100	$\sqrt{(2)}/2$	60
7	100	$\sqrt{(2)}/4$	20
8	100	$\sqrt{(2)}/4$	60

Table 16 Robustness of the results for the expected number of stock-out units

Variant	Conditions	ΔLB	ΔUB	$\Delta Length$
1	μ_1 low	+	+	+
	μ_1 high	+	+	−
2	μ_1 high and cv high	−	−	+
	Else	−	−	−
3	μ_1 low	+	+	=
	μ_1 high	+	+	−
4	μ_1 low	−	−	=
	μ_1 high	−	−	+
5	μ_1 low	=	+	+
	μ_1 high and cv high	+	+	−
	μ_1 high and cv low	+	+	+
6	μ_1 low	=	−	−
	μ_1 high and cv high	−	−	+
	μ_1 high and cv low	−	−	−

Table 17 Robustness of the results for the expected stock-out probability

Variant	Conditions	ΔLB	ΔUB	$\Delta Length$
1		+	+	+
2		−	−	−
3	μ_1 low	+	+	=
	μ_1 high and cv high	+	+	+
	μ_1 high and cv low	+	+	−
4	μ_1 low	−	−	=
	μ_1 high and cv high	−	−	−
	μ_1 high and cv low	−	−	+
5	μ_1 low	−	+	+
	μ_1 high	+	+	+
6	μ_1 low	+	−	−
	μ_1 high	−	−	−

interval depend on the uncontrollable factors of the experiment. For the first variant, only the level of μ_1 influences the results. For variant 2, both the level of μ_1 and the coefficient of variation have an impact on the results. For variants 3 and 4, the changes in the length of the interval depend on μ_1. The results for variants 5 and 6 show that an increase/decrease in the variance leads to an increase/decrease in the upper and lower bound. Only when μ_1 is low, the lower bound remains the same while the variance increases/decreases. The changes in the length of the interval depend on both μ_1 and the coefficient of variation. The increase/decrease of the upper and lower bounds is higher when μ_1 is increased/decreased (12 % on average) compared to when the variance is increased/decreased (3 % on average).

The results for the expected stock-out probability show that there is no influence of the uncontrollable factors for the first two variants. An increase/decrease of μ_1 leads to an increase/decrease in the upper and lower bounds and the length of the interval. For variants 3 and 4, the upper and lower bounds increase/decrease when μ_1 is increased/decreased. Both μ_1 and the coefficient of variation have an impact on the changes in the length of the interval. For variants 5 and 6, only μ_1 influences the changes in the bounds and the length of the interval. The results indicate that an increase/decrease of μ_1 has a higher impact (8 % on average) on the upper and lower bounds than an increase/decrease in the variance (4 % on average).

In this section, both the mean and the standard deviation of demand during lead time are increased or decreased with 10 %. These changes lead to an increase or decrease of the corresponding lower and upper bound of less than 10 % on average and of 20 % at most. For a decision-maker, this means that if there is an error in the estimates of the mean or the variance of demand during lead time after the bounds are calculated, the real bounds do not differ much from the calculated bounds. The upper and lower bounds increase/decrease if the mean or the variance increases/ decreases. Furthermore, the impact of an error in the mean is higher than the impact of an error in the variance, so for decision-makers, it is more important to have a correct estimation of the mean.

In general, it can be concluded that the bounds are robust to changes in the first and second moment of the demand distribution during lead time.

7 Conclusions and further research

Inventory on slow-moving items or products, which have been recently introduced to the market, face lack of information on the distribution of demand during lead time. The aspect of incomplete information has been addressed several times in literature, for example in the single-period (newsvendor) inventory problem. A number of papers deal with the estimation of demand uncertainty when historical demand data are not available. We address the case where insufficient data are available to decide on the functional form of the demand distribution function. Incomplete but not full information on this distribution is available like the range, the expected value, the standard deviation and/or the mode. In this case, customer service performance measures, like the expected number of stock-out units or the stock-out probability, do not lead to a unique value for the level of safety stock, but rather to an interval of values. This is due to the fact that many distributions may satisfy the incomplete information that is available. In this research, the intervals of safety stock levels are determined to support the decision-maker who has formulated targets for both performance measures. The method concentrates on the calculation of the intervals of safety stock levels with the range, the expected value and the standard deviation available as incomplete information. This information is of great use to the decision-maker to make decision on variables like the inventory level. Depending on the degree of optimism of the decision-maker, any value between the upper and lower bound on the inventory level can be used. As most textbooks advise to make use of the Normal distribution for determining the safety stock level, the results are compared to this traditional approach. But also comparisons to other then the Normal distributions have been tested. The use of these other distributions can lead to an actual number of stock-out units or an actual stock-out probability that is considerably higher than targeted. As one may question the statistical validity of the estimated parameter values, which make up the incomplete information, the robustness of the bounds of the intervals to uncertainty in the parameters is tested. Based on the results, it can be concluded that both the lower and upper bounds of the interval are robust to changes in the expected value and the standard deviation of the demand during lead time. In this research, the upper and lower bounds on the safety stock level are calculated using the range, expected value and standard deviation of demand during lead time. Additional information on the distribution like a unique mode cannot be used in the bounds developed in this paper. In further research, it might be useful to extend the results, making use of the same methodology, in case additional information would be available.

Appendix

Number of stock-out units

Upper bounds

As already stated before, the problem is to find:

$$\sup_{F \in \Phi} \int_0^b f(x)\, \mathrm{d}F(x)$$

where Φ is the class of all distribution functions with range [0,*b*] and moments μ_1 and μ_2 and where $f(x) = (x - d)_+$.

The two-point distribution that will be used depends on the position of the parameter *d* in the interval [0,*b*]. Three situations can be distinguished.

Parabola through (0,0) and (0′,f(0′)) There exists a two-point distribution with moments μ_1 and μ_2 in (0,0′). The formula for the parabola can be used with $u = 0$, $v = 0'$ and $f(0) = 0$.

$$g(x) = \frac{1}{0'^2}\left[f(0')0'x + (f'(0')0' - f(0'))x(x - 0')\right].$$

To assure that $g \geq 0$ on [0, *d*], we impose $g'(0) \geq 0$, which means that

$$f'(0') \leq \frac{2f(0')}{0'}$$

or

$$d \leq \frac{0'}{2}.$$

The best upper bound is $q_{0'}f(0')$ or

$$\frac{\mu_1}{\mu_2}(\mu_2 - \mu_1 d).$$

Parabola through (r,0) and (r′,f(r′)) The formula for the parabola is used with $v = r$, $u = r'$, $f(v) = 0$ and $f'(v) = 0$. This gives us:

$$g(x) = \frac{f(r')(x - r)^2}{(r' - r)^2}.$$

The condition $g'(u) = f'(r')$ leads to

$$2f(r') = (r' - r)f'(r')$$

or

$$d = \frac{r + r'}{2}.$$

A unique solution (r,r′) can be assured by imposing the condition

$$\frac{0'}{2} \leq d \leq \frac{b + b'}{2}.$$

Under this condition, the best upper bound is $q_{r'}f(r')$ or

$$\frac{\mu_1 - d + \sqrt{(\mu_2 - \mu_1^2) + (d - \mu_1)^2}}{2}.$$

Parabola through (b′,0) and (b,f(b)) In this case, we take $u = b$, $v = b'$, $f(v) = 0$ and $f'(v) = 0$ and obtain

$$g(x) = \frac{f(b)(x - b')^2}{(b - b')^2}.$$

To assure $g \geq f$, we impose $g'(b) \leq f'(b)$ or

$$2f(b) \leq (b - b')f'(b)$$

or

$$d \geq \frac{b + b'}{2}.$$

In that case, the upper bound is $q_b f(b)$ or

$$\frac{(\mu_2 - \mu_1^2)(b - d)}{(\mu_2 - \mu_1^2) + (b - \mu_1)^2}.$$

Lower bounds

The problem is to find:

$$\inf_{F \in \Phi} \int_0^b f(x)\, \mathrm{d}F(x)$$

where Φ is the class of all distribution functions with range [0,*b*] and moments μ_1 and μ_2 and where $f(x) = (x - d)_+$.

Here, also three situations can be distinguished, depending on the position of *d* in the interval [0,*b*].

$0 \leq d \leq b'$ A solution is found when P is the straight line through $(d, 0)$, $(\mu_1, f(\mu_1))$ and $(b, f(b))$. The three-point distribution will have masses:

$$q_d = \frac{\mu_2 - \mu_1^2}{(d - \mu_1)(d - b)};\ q_{\mu_1} = \frac{\mu_2 - \mu_1^2 + (\mu_1 - d)(\mu_1 - b)}{(\mu_1 - d)(\mu_1 - b)};\ q_b = \frac{\mu_2 - \mu_1^2}{(b - d)(b - \mu_1)}$$

The lower bound equals $q_{\mu\_1}f(\mu_1) + q_b f(b)$ or

$$\mu_1 - d.$$

$b' < d < 0'$ In this case, P is the parabola through (0, 0), (*d*,0) and (*b*, *f*(*b*)). The best lower bound is $q_b f(b)$ or

$$\frac{\mu_2 - \mu_1 d}{b}.$$

$0' \leq d \leq b$ Here, a solution is found when P is the straight line through (0, 0), (μ, 0) and (*d*,0). The best lower bound is equal to 0.

Stock-out probability

When calculating bounds on tail probabilities [2], the problem is to find:

$$\sup_{F\in\Phi}\int_0^b f(x)\,\mathrm{d}F(x)$$

and

$$\inf_{F\in\Phi}\int_0^b f(x)\,\mathrm{d}F(x)$$

where Φ is the class of all distribution functions with range $[0,b]$ and with moments μ_1 and μ_2 known and where

$$f(x) = \begin{cases} 0 & \text{if } x \le d; \\ 1 & \text{if } x > d. \end{cases}$$

Upper bounds

$0 \le d \le b'$ A solution is found when P is the straight line through $(b',1)$ and $(b,1)$. The upper bound is equal to $q_{b'}f(b') + q_b f(b) = 1$.

$b' < d \le 0'$ In this case, P is the parabola through (0, 0), $(d, 1)$ and $(b,1)$. According to Lemma 2, the three-point distribution in $(0, d, b)$ will have masses:

$$q_d = \frac{b\mu_1 - \mu_2}{d(b-d)}, q_b = \frac{\mu_2 - \mu_1 d}{b(b-d)}, q_0 = 1 - q_d - q_b.$$

The upper bound is $q_d f(d) + q_b f(b)$ or

$$\frac{(b+d)\mu_1 - \mu_2}{bd}.$$

$0' < d \le b$ Here, the solution is the parabola through $(d',0)$ and $(d,1)$ and tangent to $f(x)$ in d'. The best upper bound is $q_d f'(d) + q_{d'} f(d')$ or

$$\frac{\mu_2 - \mu_1^2}{\mu_2 - \mu_1^2 + (\mu_1 - d)^2}.$$

Lower bounds

$0 \le d \le b'$ In this case, P is the parabola through $(d,0)$ and $d',1)$ and tangent to f(x) at d'. The lower bound equals $q_d f(d) + q_{d'} f(d')$ or

$$\frac{(\mu_1 - d)^2}{\mu_2 - \mu_1^2 + (\mu_1 - d)^2}.$$

$b' < d \le 0'$ A solution is found when P is the parabola through (0, 0), $(d,0)$ and $(b,1)$. The masses of the three-point distribution give a lower bound of

$$\frac{\mu_2 - \mu_1 d}{b(b-d)}.$$

$0' < d \le b$ Here, P is the line through (0,0) and $(0',0)$. The lower bound is 0.

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