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ORIGINAL PAPER

Quantifying the impact of demand substitution on the bullwhip effect in a supply chain

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Abstract In a supply chain, the distorted demand information when it goes upstream is commonly known as the bullwhip effect. In this paper, the impact of demand substitution on the bullwhip effect in a two-stage supply chain is investigated. In our model, a single retailer observes inventory levels for two products, among which product 1 can be used to substitute product 2. The retailer places orders to a single manufacturer following an order-up-to inventory policy and uses a simple moving average forecasting method to estimate the lead-time demand. The customers' demands are modeled by an autoregressive process. By analyzing the bullwhip effect in such settings, quantitative relations between the bullwhip effect and the forecasting method, lead time, demand process, and the product substitution are obtained. Numerical results show that demand substitution can reduce the bullwhip effect in most cases.

Keywords Bullwhip effect · Demand substitution · Supply chain management

1 Introduction

A simple two-stage supply chain typically consists of manufacturers, retailers, and end customers. Only retailers have direct information of end customers' demands. As to manufacturers, only orders from retailers can be seen. End customers' demands are not visible to manufactures. Products are distributed downward along the supply chain, while information flows upward from end customers to manufacturers. It has been noticed that the information is distorted due to various reasons. This distortion of the demand in the upstream of a supply chain is widely known as the bullwhip effect [5, 12], which has been studied from both design and operation perspectives.

Figure 1 illustrates the bullwhip effect in a three-stage supply chain that consists of manufacturers, distributors, retailers, and end customers. The order quantity, placed by the distributors to manufacturers, is distorted dramatically comparing with the real end customers' demand. Because of observing only immediate order data, the entities in the supply chain are misled by the amplified demand patterns. This distorted information causes inefficiencies in many parts of the supply chain, such as excess raw materials due to unplanned purchases from suppliers, additional manufacturing expenses created by excess order demands, inefficient utilization and overtime, excessive warehouse cost, and so on. Fuller et al. [9] pointed out that the inefficiencies led by distorted information born some responsibility for $75 billion to $100 billion out of $300 billion grocery sales trapped in the pipeline as the form of inventory in 1993.

The bullwhip effect has been commonly and conveniently measured as the ratio of order variance to demand variance. It depends on the demand process at the end customer level, the order lead time, the demand forecasting model used, and the replenishment policy. In this paper, we mainly discuss the impact of demand substitution.

There are two fundamental forms of substitution: "*supplier driven*" and "*customer driven*". In supplier-driven pattern, supplier substitutes products of type A with

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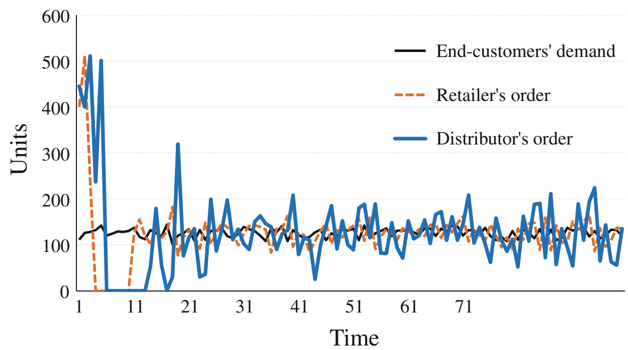

Fig. 1 The distortion of demand information

products of type B based on its own inventory position, the market forecasting, and other related information. For example, a computer manufacturing finds that 2G memory has too many inventories but 1G memory is out of stock. New shipments of 1G memory will arrive in 3 months without incurring expedited cost. In this case, the supplier can run a promotion to lower the price of 2G memory, so that part of the customers who want 1G memory will switch to 2G. The supplier-driven substitution often occurs in multi-product manufacturing system and some service industry.

In customer-driven pattern, the product substitution decision is made by the customers. When a product is stock-out, the customer either chooses other products or leaves without buying. Due to the heterogeneous customer demands, the supplier is unable to predict the decision of each customer. To prevent product from stocking out, the seller must consider all customers' interests to place the inventory replenishment order. This paper only considers the case of customer-driven substitution.

The existence of the demand substitution will finally affect the inventory control policy of items either served as possible substitutions or being substituted. For example, the inventory of items of type A can be used to satisfy the unmet demand of items of type B and thus to prevent the loss of sales. Under this circumstance, it is not necessary to carry out as much safety stock as if demands are totally independent. The demand substitution creates interdependency among items because each demand for currently unavailable item is transferred to demand for currently in stock substitutes. Optimizing the inventory policy subjected to the influence of the interdependency is a very complex problem. But it is clear that the demand substitution has significant influence on the inventory policy [1].

The rest of this paper is organized as follows. Section 2 extensively reviews relevant literature on the bullwhip effect and demand substitution. Section 3 develops the model to study the impact of demand substitution on the bullwhip effect in a simple two-stage supply chain. Section 4 presents numerical experiments and discussion of the results. Finally, Sect. 5 concludes this paper and outlines the direction of future research.

2 Literature review

The bullwhip effect was first proposed and studied by Forrester [8]. This paper laid the foundation for the research in this field. As mentioned in Chen et al. [4], the focus of most previous research has been on the following areas: (1) demonstrating the existence of the bullwhip effect, (2) identifying possible causes of the bullwhip effect, and (3) developing strategies to reduce the impact of the bullwhip effect.

Metters [16] established an empirical lower bound on the profitability to identify the magnitude of the problem caused by the bullwhip effect. He modeled the supply chain as a periodic, time-varying, stochastic demand dynamic program with capacitated production and used dynamic programming to determine the optimal ordering policy.

Lee et al. [12, 13] analyzed four important causes for the bullwhip effect, including demand signal processing, the rationing game, order batching, and price fluctuation. In addition, they proposed actions to mitigate the detrimental impact of the bullwhip effect.

Chen et al. [4] investigated the bullwhip effect in a two-stage supply chain and examined two factors that are assumed to cause the bullwhip effect, demand forecasting, and order lead time. Their work showed that centralizing demand information can reduce, but not completely eliminate, the bullwhip effect. Chen et al. [5] explored the impact of different forecasting methods on the bullwhip effect in a two-stage supply chain. And they found that using the exponential smoothing forecasting method led to a bigger bullwhip effect than using the moving average forecasting method. They concluded that by choosing proper demand process and/or forecasting method, the bullwhip effect may be reduced.

Demand substitution has been a research topic for decades. To the best of our knowledge, McGillivray and Silver [15] are the first one who studied the demand substitution. They investigated the effects of demand substitution on inventory policy and inventory/shortage costs. By studying a two-item case, they found that the inventory cost was reduced significantly, thanks to the demand substitution. In addition, they pointed out that under demand substitution, the optimal stocking rule was substantially different from the case where the two types of items were independent.

Similar to McGillivray and Silver [15], some of the research on demand substitution is also conducted in a simple two-item case. Parlar and Goyal [18] modeled the two-substitutable-product problem as an extension of a single-period inventory problem to which the result of classical news-vendor problem can be applied. They proved that the expected profit function was strictly concave for a wide range of parameters values. Rajaram and Tang [19] analyzed the impact of demand substitution on order quantities and expected profits in an extended news-vendor model. In their model, the demand of the products in shortage can be substituted by products in stock with a certain probability. They studied the mechanism that the demand uncertainty and degree of substitution affected the order quantities and expected profits. Drezner et al. [7] investigated the demand substitution effect on an economic order quantity (EOQ) model with two types of products. Three cases were studied: no substitution, full substitution, and partial substitution. The author argued that in a deterministic setting with proportional substitution cost, the full substitution could not be optimal, and the partial substitution was optimal when the transfer cost from product 1 to product 2 satisfied a certain condition; otherwise, the no substitution policy would be optimal.

Some of the research studied the situation with multiple types of products. Bitran and Dasu [3] and Bassok et al. [2] considered the demand substitution with multiple types of products under the "one-way substitutability" scenario. They divided the products into several grades. The products in the higher grades can be used to substitute the product in the lower grade with a certain cost. In their model, different types of products have different associated holding, shortage, and salvage costs. Bitran and Dasu [3] examined the demand substitution in the semiconductor industry. In their model, product demands were deterministic, but the actual quantity produced was different from the quantity being processed due to random yield of products. Then, they extended their research with random demand for products and adding setup cost for each product substitution. Bassok et al. [2] considered a single-period multi-product inventory problem with substitution and proportional costs and revenues. Their study concluded that the benefit of demand substitution was higher when the demand variability and salvage values of products were high and substitution cost and profit margins were low. Drezner and Gurnani [6] extended their previous research [7] from two products to N products. They studied a deterministic nested substitution problem in which multiple products could be substituted for each other at a certain cost under an EOQ setup. Their research was also conducted under a "one-way substitutability" scenario. Agrawal and Smith [1] considered the problem of optimizing assortments in a multi-item retail inventory system. The customers bought items in set. If one item was not available, the customer either walked away or accepted a substitution or change the purchased item set. A demand model to capture this behavior was proposed to derive a tractable approximation of the problem. By assuming a fixed cycle for replenishment with no lead time, their model reduced to a multi-item news-vendor model. Netessine and Rudi [17] considered demand substitution problem with multi-products in two different scenarios in the demand substitution: the centralized management where all products were managed by a central decision maker and decentralized inventory management where each product was managed by an independent decision maker. In their model, a deterministic proportion of unsatisfied demand for a product can be substituted by other type of products.

Some researchers extended their study on demand substitution to field of service industry, represented by Karaesmen and van Ryzin [11]. They studied an overbooking problem with multiple reservation and inventory classes. Different from classical revenue management problem, the inventory classes may be used as a substitution to satisfy the demand of a given reservation class. The object was to maximize the expected profit and determine overbooking levels for the reservation classes by taking substitution as an option.

Although there are plenty of studies on the bullwhip effect and demand substitution, to the best of our knowledge, we are the first one who examine the impact of demand substitution on the bullwhip effect and identify the potential opportunities to mitigate the bullwhip effect by considering demand substitution.

3 Problem definition and model development

The bullwhip effect is commonly known as the propagation and amplification of order fluctuations from lower level of a supply chain to its upper level. We are interested in the impact of demand substitution on the bullwhip effect. In our model, we investigate this impact in a simple supply chain that includes a single retailer, a single manufacturer, and two types of products. By calculating the ratio of variance of retailer's orders to the manufacturers to the variance of demands from the end customers, we are able to quantify the bullwhip effect.

3.1 Notations

Let us define the following notations:

- i index for products, $i = 1, 2$;
- t index for time period number;
- λ percentage of product 1 can be used to substitute product 2, $0 \leq \lambda < 1$;
- L the lead time;
- p the number of demand observation periods used in the moving average;
- μ_i average demand of product i in the autoregressive demand model;

ρ_i the autocorrelation coefficient of the autoregressive model of product i;
$\varepsilon_{t,i}$ forecast error for product i during time period t;
$D_{t,i}$ demand for product i during time period t;
$D^L_{t,i}$ lead-time demand for product i;
$\hat{D}^L_{t,i}$ the forecast of the lead-time demand for product i;
z_i normal z-score, determined by the desired service level;
$\hat{\sigma}^L_{t,i}$ standard deviation of forecast error of lead-time demand of product i;
$y_{t,i}$ order-up-to level for product i at the beginning of time period t;
$q_{t,i}$ order quantity for product i at the beginning of time period t;
LB_i the lower bound of the bullwhip effect of product i;
$C_{L,\rho}$ a constant function of L and ρ

3.2 Demand process

We make the following assumptions in our problem. At the beginning of time period t, the retailer estimates the demands for two products during time period t, $D_{t,i}$, following the moving average forecasting scheme. Suppose the order cost is negligible, and the retailer places an order, $q_{t,i}$, to the manufacturer according to the order-up-to policy. Further, we assume that a fixed percentage, λ, of product 1 is used to substitute product 2.

It is supposed that end customers' demands are modeled by an autoregressive (AR) demand process that has been applied to the analysis of the bullwhip effect and information sharing by many researchers such as Zhang [21] and Gilbert [10]. Taking demand substitution into consideration, the end customers' demands for the two products during time period t can be given as follows:

$$\begin{aligned} D_{t,1} &= \mu_1 + \rho_1 D_{t-1,1} + \epsilon_{t,1} + \lambda D_{t,1} \\ D_{t,2} &= \mu_2 + \rho_2 D_{t-1,2} + \epsilon_{t,2} - \lambda D_{t,1}, \end{aligned} \tag{1}$$

where $D_{t,i}$ is the end customer demand for product i in period t, μ_i is the average demand, ρ_i represents the autocorrelation parameter with $|\rho_1| < 1 - \lambda$ and $|\rho_2| < 1$, and $\epsilon_{t,i}$ are independent and identically distributed (i.i.d.) from a symmetric distribution with mean 0 and standard deviation σ_i, $i = 1, 2$.

From (1), the following results can be obtained.[1]

$$E(D_{t,1}) = \frac{\mu_1}{1 - \rho_1 - \lambda} \tag{2a}$$

$$\mathrm{Var}(D_{t,1}) = \frac{\sigma_1^2}{(1-\lambda)^2 - \rho_1^2} \tag{2b}$$

$$E(D_{t,2}) = \frac{\mu_2(1 - \rho_1 - \lambda) - \lambda\mu_1}{(1 - \rho_1 - \lambda)(1 - \rho_2)} \tag{2c}$$

$$\mathrm{Var}(D_{t,2}) = \frac{[(1-\lambda)^2 - \rho_1^2]\sigma_2^2 - \lambda^2\sigma_1^2}{[(1-\lambda)^2 - \rho_1^2](1 - \rho_2^2)}. \tag{2d}$$

3.3 Inventory policy

In our model, we assume that the retailer follows a simple *order-up-to* inventory policy to bring the actual inventory level, $y_{t-1,i} - D_{t-1,i}$, to the target inventory level, $y_{t,i}$. We assumed the lead times, defined as the delay between placing an order to receiving the order, for both products are the same, fixed L periods. In other words, the order placed at the start of period t is received at the start of period $t + L$. The order quantities for the two products at the beginning of time period t can be determined as

$$\begin{aligned} q_{t,1} &= y_{t,1} - (y_{t-1,1} - D_{t-1,1}) \\ &= y_{t,1} - y_{t-1,1} + D_{t-1,1} \quad \text{Product 1} \\ q_{t,2} &= y_{t,2} - (y_{t-1,2} - D_{t-1,2}) \\ &= y_{t,2} - y_{t-1,2} + D_{t-1,2} \quad \text{Product 2} \end{aligned}$$

Note that $q_{t,i}$ might be negative if the remaining inventory from period $t - 1$ is greater than the order-up-to level of period t. In this case, it is treated as excess inventory and can be returned without any cost. Thus, our model is simplified without considering inventory holding cost. Chen et al. [5] proved that $\mathrm{Var}(q)$ and $\mathrm{Var}(q^+)$ were quite close. Here $q^+ = \max\{q_t, 0\}$.

The target inventory $y_{t,i}$ at the beginning of period t is estimated from the observed demand as

$$\begin{aligned} y_{t,1} &= \hat{D}^L_{t,1} + z_1\hat{\sigma}^L_{t,1} \\ y_{t,2} &= \hat{D}^L_{t,2} + z_2\hat{\sigma}^L_{t,2}, \end{aligned} \tag{3}$$

where $\hat{D}^L_{t,i}$ is the forecast of lead-time demand for product i, $\hat{\sigma}^L_{t,i}$ is the standard deviation of the forecast error of lead-time demand for product i, z_i is the normal z-score that chosen to meet the desired service level of the inventory policy, and $z_i\hat{\sigma}^L_{t,i}$ estimates safety inventory during the lead time.

We suppose that the retailer uses a simple moving average (MA) to estimate $D^L_{t,i}$ and $\sigma^L_{t,i}$ based on the information of the past p periods. Both products are assumed to have the same predication periods, i.e., p is the same to both products. According to MA,

$$\hat{D}^L_{t,i} = L\left(\frac{\sum_{j=1}^{p} D_{t-j,i}}{p}\right) \tag{4}$$

$$\hat{\sigma}^L_{t,i} = C_{L,\rho}\sqrt{\frac{\sum_{j=1}^{p}(D_{t-j,i} - \hat{D}_{t-j,i})^2}{p}}, \tag{5}$$

[1] The explicit technique details are shown in the Appendix.

where $D_{t-j,i} - \hat{D}_{t-j,i}$ is the forecast error of the $(t-j)^{th}$ period of product i, and $C_{L,\rho}$ is a constant function of L, ρ and p. A detailed discussion of this constant is given by Ryan [20].

To quantify the bullwhip effect under demand substitution, we should determine how the variance of $q_{t,i}$ is relative to the variance of $D_{t,i}$. In other words, the quantified bullwhip effect of this simple supply chain in period t is $\frac{\mathrm{Var}(q_{t,i})}{\mathrm{Var}(D_i)}$.

3.4 Quantifying the bullwhip effect

Since the variance of $q_{t,i}$ and D_i are different for the two products, the bullwhip effect of the two products will be discussed separately.[2]

Given the equations of the order-up-to level, demand forecasting, and standard deviation of forecast error, we can further express $q_{t,1}$ as follows:

$$\begin{aligned} q_{t,1} &= y_{t,1} - y_{t-1,1} + D_{t-1,1} \\ &= (\hat{D}^L_{t,1} + z_1\hat{\sigma}^L_{t,1}) - (\hat{D}^L_{t-1,1} + z_1\hat{\sigma}^L_{t-1,1}) + D_{t-1,1} \\ &= (1 + L/p)D_{t-1,1} - (L/p)D_{t-p-1,1} + z_1(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}), \end{aligned}$$

Then, the variance of the order quantity $q_{t,1}$ for product 1 at time period t is as follows:

$$\begin{aligned} \mathrm{Var}(q_{t,1}) &= \mathrm{Var}[(1 + L/p)D_{t-1,1} - (L/p)D_{t-p-1,1} \\ &\quad + z_1(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1})] \\ &= \mathrm{Var}(D_1)\left[1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\left(1 - \frac{\rho_1^p}{(1-\lambda)^p}\right)\right] \\ &\quad + z_1^2 Var(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}). \end{aligned} \tag{6}$$

For product 2, we apply the same procedures as product 1.

$$\begin{aligned} q_{t,2} &= y_{t,2} - y_{t-1,2} + D_{t-1,2} \\ &= (\hat{D}^L_{t,2} + z_1\hat{\sigma}^L_{t,2}) - (\hat{D}^L_{t-1,2} + z_1\hat{\sigma}^L_{t-1,2}) + D_{t-1,2} \\ &= (1 + L/p)D_{t-1,2} - (L/p)D_{t-p-1,2} + z_2(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}), \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(q_{t,2}) =& \mathrm{Var}[(1 + L/p)D_{t-1,2} \\ & - (L/p)D_{t-p-1,2} + z_2(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2})] \\ =& \left(1 + 2\frac{L}{p} + 2\frac{L^2}{p^2}\right)\mathrm{Var}(D_2) \\ & - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\rho_2^p \mathrm{Var}(D_2) \\ & - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\frac{\lambda^2(1-\lambda)\mathrm{Var}(D_1)}{(1-\lambda-\rho_1\rho_2)} \\ & \times \sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right) + z_2^2 Var(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}). \end{aligned} \tag{7}$$

Given the above derivation, the bullwhip effects for product 1 and product 2 in period t are given below:

$$\frac{\mathrm{Var}(q_{t,1})}{\mathrm{Var}(D_1)} \geq 1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\left(1 - \frac{\rho_1^p}{(1-\lambda)^p}\right), \tag{8}$$

$$\begin{aligned} \frac{\mathrm{Var}(q_{t,2})}{\mathrm{Var}(D_2)} \geq& 1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)(1 - \rho_2^p) \\ & - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\frac{\lambda^2(1-\lambda)}{(1-\lambda-\rho_1\rho_2)}\frac{\mathrm{Var}(D_1)}{\mathrm{Var}(D_2)} \\ & \times \sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right), \end{aligned} \tag{9}$$

the bounds are tight when $z_1 = 0$, $z_2 = 0$, that is, the security inventory is not taken into considerations [14].

The lower bound of the bullwhip effect of product i has no relation with time period. Therefore, we use LB_i to denote the lower bounds of the bullwhip effect of product i, omitting the subscript of time periods.

$$\mathrm{LB}_1 = 1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\left(1 - \frac{\rho_1^p}{(1-\lambda)^p}\right) \tag{10}$$

$$\begin{aligned} \mathrm{LB}_2 =& 1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)(1 - \rho_2^p) \\ & - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\frac{\lambda^2(1-\lambda)}{(1-\lambda-\rho_1\rho_2)}\frac{\mathrm{Var}(D_1)}{\mathrm{Var}(D_2)} \\ & \times \sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right). \end{aligned} \tag{11}$$

4 Numerical experiments

Several observations can be made from (10) and (11). First, we notice that LB_1 and LB_2 are functions of the following parameters: (a) p, number of observations used in MA, (b) L, the lead time, (c) ρ_1 and ρ_2, first-order autocorrelation coefficients of the autoregressive demand process of product 1 and product 2, respectively, and (d) λ, the substitution percentage. In the rest of this section, we will discuss the influence of the five parameters on the lower bounds of the bullwhip effect for both products.

4.1 No demand substitution

If $\lambda = 0$, i.e., product 1 is not used to substitute product 2, LB_1 and LB_2 have the same format.

$$\mathrm{LB}_1 = 1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)(1 - \rho_1^p)$$

$$\mathrm{LB}_2 = 1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)(1 - \rho_2^p)$$

There are substantial literatures investigating this problem. Chen et al. [4] pointed out that the variability amplification

[2] The explicit technique details are shown in the Appendix.

of product demand (the bullwhip effect) was (a) a decreasing function of p, the number of observations used in MA, (b) an increasing function of L, the lead time, (c) a decreasing function of ρ, when $\rho > 0$, and (d) larger for odd values of p than for even values of p, when $\rho < 0$.

Let us put aside the equations and think about the problem intuitively. The result of the study shows that (a) the smoother the demand forecasts, the smaller the increase in the bullwhip effect; (b) the longer the lead times, the more demand data are needed to reduce the bullwhip effect; (c) the higher the degree of the demands is positively correlated, the smaller the increase in variability. All these conclusions are in line with our experience.

If product 1 is used to substitute product 2, i.e., $\lambda > 0$, the situation is very complex. We will discuss the influence of λ on LB_1 and LB_2, separately.

4.2 The effect of demand substitution on LB_1

Figures 2, 3, and 4 show the effect of λ on the lower bound of the bullwhip effect when the number of the observation used in MA p, the order lead time L, and the autoregressive coefficient of product 1 ρ_1 are changed, respectively. When $\lambda > 0$, i.e., with demand substitution, similar to the conclusion in Sect. 4.1, the lower bound of the bullwhip effect of product 1 is still a decreasing function of p and increasing function of L. The impact of demand substitution on lower bound of the bullwhip effect of product 1 is correlated with the impact of the autoregressive coefficient of product 1 ρ_1 and the parity of p.

When $\rho_1 > 0$, the lower bound of the bullwhip effect of product 1 is a decreasing function of λ no matter what value of the other two parameters are. Technically, we can

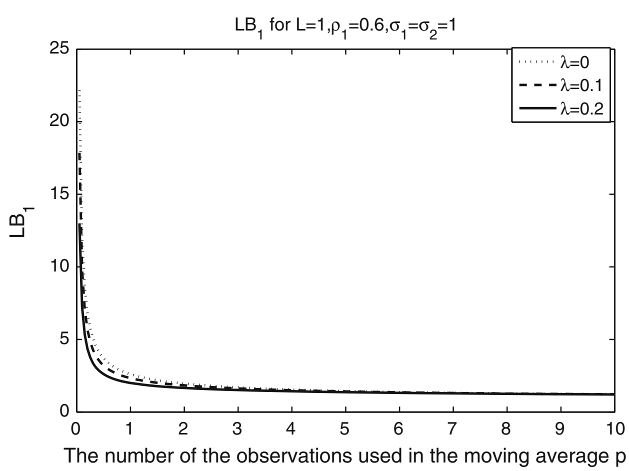

Fig. 2 Impact of number of the observation used in MA on the lower bound of the bullwhip measure of product 1. It shows that the lower bound of the bullwhip effect of product 1 is a decreasing function of p and the demand substitution can reduce the lower bound of the bullwhip effect since the curve is lower for $\lambda > 0$ than $\lambda = 0$

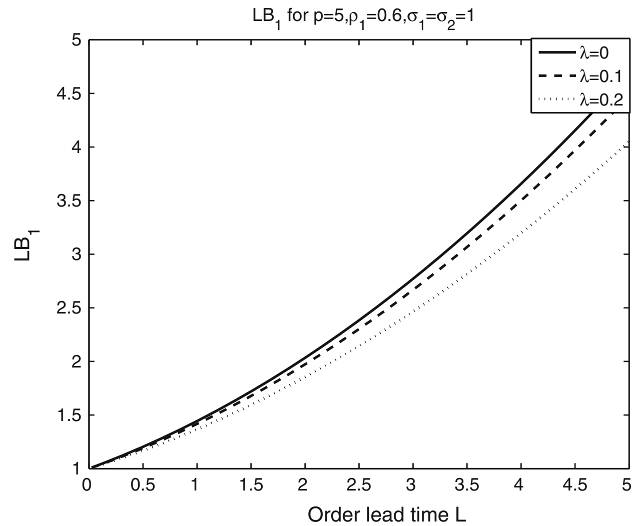

Fig. 3 Impact of order lead time on the lower bound of the bullwhip effect of product 1. It shows that the lower bound of the bullwhip effect of product 1 is an increasing function of L and the demand substitution can reduce the lower bound of the bullwhip effect since the curve is lower for $\lambda > 0$ than $\lambda = 0$

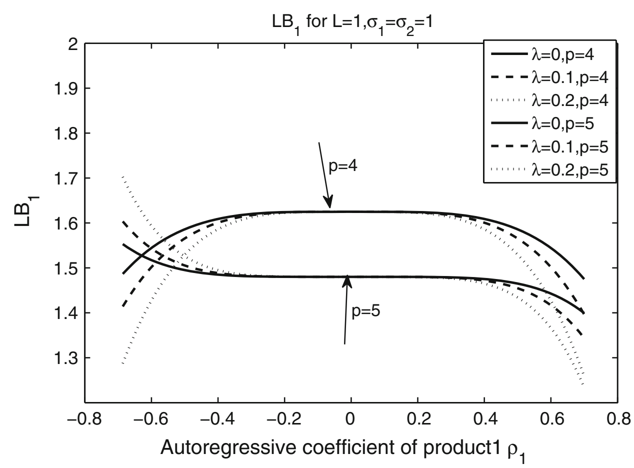

Fig. 4 Impact of autoregressive coefficient of product 1 on the lower bound of the bullwhip effect of product 1. For even value of p, the lower bound of the bullwhip effect of product 1 is a decreasing function of $|\rho_1|$ and the demand substitution makes contribution on reducing the lower bound of the bullwhip effect of product 1. This is true for odd value of p when ρ_1 is positive. However, when p is odd and ρ_1 is negative, the lower bound of the bullwhip effect increases when $|\rho_1|$ increases. And under this situation, the demand substitution will cause increase in the bullwhip effect

also draw this conclusion according to (10). Specifically, if we change the expression of $D_{t,1}$ into:

$$D_{t,1} = \frac{\mu_1}{\lambda} + \frac{\rho_1}{\lambda} D_{t-1,1} + \frac{\epsilon_{t,1}}{\lambda}, \tag{12}$$

ρ_1/λ can be seen as autoregressive coefficient of the demand process of product 1. For $\rho_1 > 0$, if λ increases,

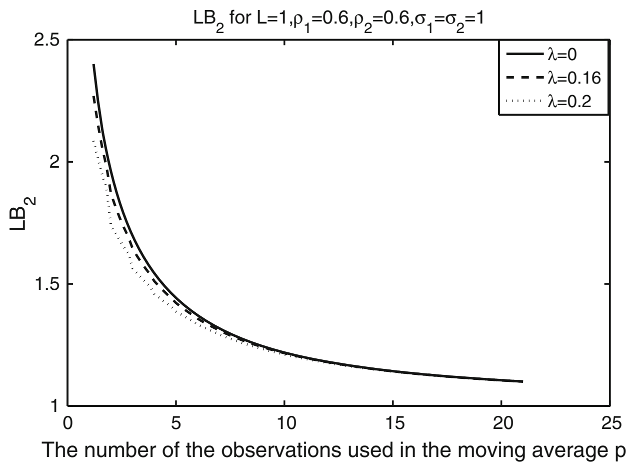

Fig. 5 Impact of number of the observation used in MA on the lower bound of the bullwhip effect of product 2. It shows that the lower bound of the bullwhip effect of product 2 is an increasing function of L and the demand substitution can reduce the lower bound of the bullwhip effect since the curve is lower for $\lambda > 0$ than $\lambda = 0$

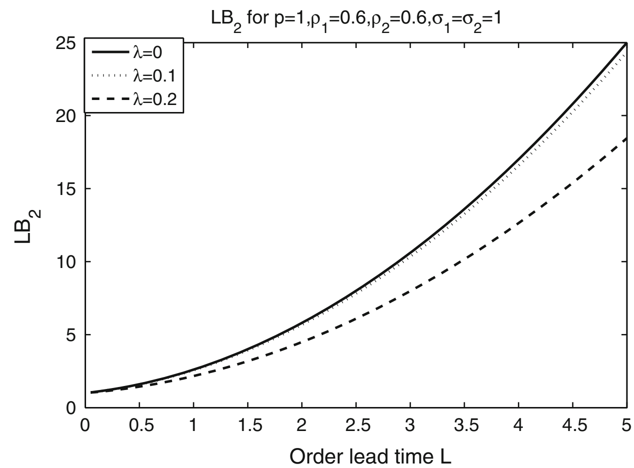

Fig. 6 Impact of order lead time on the lower bound of the bullwhip effect of product 2. It shows that the lower bound of the bullwhip effect of product 2 is an increasing function of L and the demand substitution can reduce the lower bound of the bullwhip effect since the curve is lower for $\lambda > 0$ than $\lambda = 0$

ρ_1/λ will decrease. In other words, the correlation between successive period demands of product 1 decreases. Therefore, the variability amplification of demand for product 1 will increase.

However, when $\rho_1 < 0$, i.e., successive period demands are negatively correlated, we observe interesting behavior. When p is even, LB_1 is an increasing function of ρ_1, and the demand substitution plays a positive role in reducing the lower bound of the bullwhip effect of product 1. For odd values of p, LB_1 is a decreasing function of ρ_1, and LB_1 is larger for $\lambda > 0$ than for $\lambda = 0$. In other words, when the successive period demands are negative related and the number of observation used in MA is odd, the demand substitution leads to increase in the lower bound of the bullwhip effect of product 1. In addition, when ρ_1 is negative and smaller than a certain value, the lower bound of the bullwhip effect of product 1 will be larger for odd values of p than for $p - 1$.[3]

4.3 The effect of demand substitution on LB_2

Figures 5, 6, 7, 8, 9, and 10 show this effect of λ on the lower bound of the bullwhip effect of product 2 when the number of the observation used in MA p, the order lead time L, and the autoregressive coefficients ρ_1 and ρ_2 are changed, respectively. When $\lambda > 0$, similar to the conclusion in Sect. 4.1 and 4.2, the lower bound of the bullwhip effect of product 2 is still a decreasing function of p and increasing function of L. The impact of demand

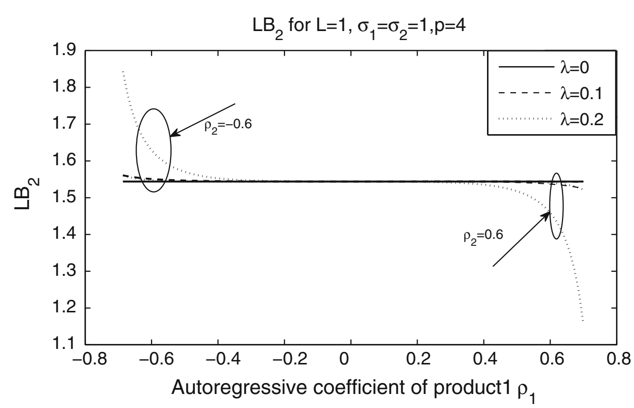

Fig. 7 Impact of autoregressive coefficient of product 1 on the lower bound of the bullwhip effect of product 2. For even value of p, when ρ_2 is positive, the lower bound of the bullwhip effect of product 2 is a decreasing function of ρ_1 and the demand substitution makes contribution on reducing the bullwhip effect. When ρ_2 is negative, the lower bound of the bullwhip effect will increase as $|\rho_1|$ increasing and the demand substitution will cause bigger amplification of the demand

substitution on lower bound of the bullwhip effect of product 2 is correlated with the impacts of autoregressive coefficient ρ_1 and ρ_2 as well as the parity of p.

For odd values of p, the demand substitution always makes contribution on reducing the bullwhip effect of product 2. Specifically, the lower bound of the bullwhip effect of product 2 is a concave function of ρ_1. In addition, LB_2 is a decreasing function of λ, and for $\rho_1\rho_2 > 0$, the demand substitution causes more quickly decrease in the lower bound of the bullwhip effect of product 2 along with the increase of $|\rho_1|$. The lower bound of the bullwhip effect of product 2 is a convex function of ρ_2 when $\rho_2 < 0$ and a concave function of ρ_2 when $\rho_2 > 0$.

[3] Chen et al (2000a) oversimplified the situation to claim that for $\rho < 0$ the lower bound on the increase in variability will be large for odd values of p than for even values of p.

For even values of p, sometimes the demand substitution will cause increase in the lower bound of the bullwhip effect of product 2. Specifically, the lower bound of the bullwhip effect of product 2 is a decreasing function of ρ_1 when ρ_2 is positive, and the demand substitution makes contribution on reducing the bullwhip effect. When ρ_2 is negative, the lower bound of the bullwhip effect will increase as $|\rho_1|$ increasing and the demand substitution will cause bigger amplification of the demand. The lower bound of the bullwhip effect of product 2 is a concave function of ρ_2. In addition, only when $\rho_1 > 0$, the demand substitution can reduce the lower bound of the bullwhip effect of product 2. However, when $\rho_1 < 0$, the demand substitution will cause increase in the bullwhip effect. In technical level, we can prove that when $\frac{\rho_1}{1-\lambda} + \rho_2 < 0$, for even values of p, the demand substitution will increase the lower bound of the bullwhip effect of product 2. We give the proof as follows.

When $p = 2n$, $n = 1, 2, 3, \ldots,$

$$\sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right) = \frac{\rho_2^0\rho_1^{p+1}}{(1-\lambda)^{p+1}} + \frac{\rho_2^1\rho_1^p}{(1-\lambda)^p} + \cdots + \frac{\rho_2^{p-2}\rho_1^3}{(1-\lambda)^3}\frac{\rho_2^{p-1}\rho_1^2}{(1-\lambda)^2}$$

$$= \frac{\rho_2^0\rho_1^p}{(1-\lambda)^p}\left(\frac{\rho_1}{1-\lambda} + \rho_2\right) + \cdots + \frac{\rho_2^{p-2}\rho_1^2}{(1-\lambda)^2}\left(\frac{\rho_1}{1-\lambda} + \rho_2\right)$$

$$= \left(\frac{\rho_1}{1-\lambda} + \rho_2\right)\left(\frac{\rho_2^0\rho_1^p}{(1-\lambda)^p} + \cdots + \frac{\rho_2^{p-2}\rho_1^2}{(1-\lambda)^2}\right).$$

Note that for even values of p

$$\frac{\rho_2^0\rho_1^p}{(1-\lambda)^p} + \cdots + \frac{\rho_2^{p-2}\rho_1^2}{(1-\lambda)^2} > 0.$$

So for $\frac{\rho_1}{1-\lambda} + \rho_2 < 0$,

$$\sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right) < 0.$$

In addition, $\frac{\mathrm{Var}(D_1)}{\mathrm{Var}(D_2)} > 0$ and $1 - \lambda - \rho_1\rho_2 > 0$ (since $|\rho_1| < 1 - \lambda$), so

$$\left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\frac{\lambda^2(1-\lambda)}{(1-\lambda-\rho_1\rho_2)}\frac{\mathrm{Var}(D_1)}{\mathrm{Var}(D_2)}\sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right) < 0.$$

So in this case, the demand substitution increases the lower bound of the bullwhip effect of product 2. This conclusion provides a criterion for making decision on demand substitution. If it is ignored, the demand substitution will make the situation worse.

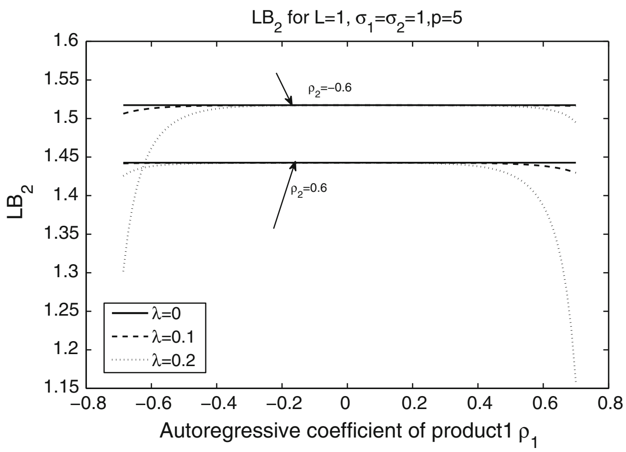

Fig. 8 Impact of autoregressive coefficient of product 1 on the lower bound of the bullwhip effect of product 2. For odd value of p, no matter ρ_2 is positive or not, the lower bound of the bullwhip effect of product 2 is a decreasing function of $|\rho_1|$. Moreover, for $\rho_1\rho_2 > 0$, the demand substitution causes more quickly decrease in the lower bound of the bullwhip effect

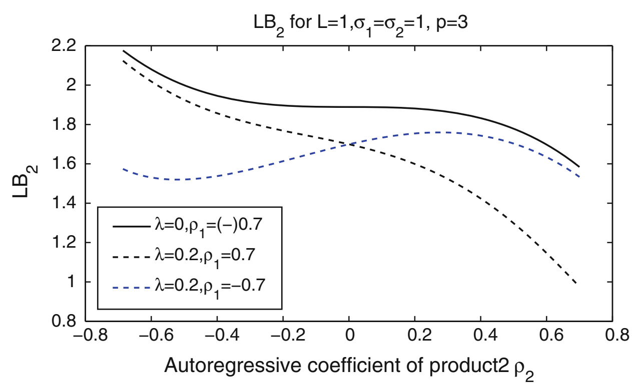

Fig. 9 Impact of autoregressive coefficient of product 2 on the lower bound of the bullwhip effect of product 2. For odd value of p, the demand substitution can make contribution on reducing the lower bound of the bullwhip effect of product 2. Furthermore, for $\rho_1\rho_2 > 0$, the demand substitution causes faster decrease in the lower bound of the bullwhip effect along with the increase of $|\rho_2|$

5 Discussion and conclusions

In this research, we quantified the impact of the demand substitution on the bullwhip effect in a simple supply chain with a single retailer, a single manufacturer, and two types of products. And product 1 can be used to substitute product 2. It is assumed that the customer's demands are forecasted by first-order autoregressive demand process. The retailer employs order-up-to inventory policy and follows a simple MA forecasting method to estimate the lead-time demand. By quantifying the bullwhip effect in such assumption, we obtain relations between the bullwhip effect and the demand variance, order policy as well as the product substitution.

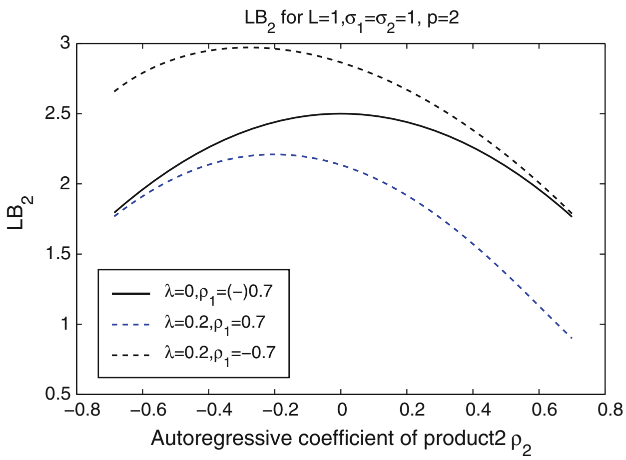

Fig. 10 Impact of autoregressive coefficient of product 2 on the lower bound of the bullwhip effect of product 2. For even value of p, only when $\rho_1 > 0$, the demand substitution can reduce the lower bound of the bullwhip effect of product 2. And roughly speaking, along with the increase of ρ_2, the demand substitution causes faster decrease in the lower bound of the bullwhip effect. However, when $\rho_1 < 0$, the demand substitution will cause increase in the bullwhip effect

The lower bounds of the bullwhip effect for both products are decreasing function of p, the number of observations used to estimate the mean and the variance of demand. Figures 2 and 5 show the relationship between the lower bounds of the bullwhip effect and the smoothing periods (p). When p is large, the lower bound of the bullwhip effect of both product is really small. However, when p is small, the lower bound of the bullwhip effect is extremely big. In summary, the smoother the demand forecasts, the smaller the increase in the lower bound of the bullwhip effect.

The lower bounds of the bullwhip effect of both products are increasing functions of L, the lead-time parameter. Figures 3 and 6 show the relationship between the lower bounds of the bullwhip effect and the lead time L. As can be seen from both Eq. 10 and Eq. 11, the larger L, the higher the lower bounds are. If the lead time is doubled, to maintain the same order of the lower bound of the bullwhip effect, twice demand data must be supplied. That is, the retailer must use more demand data in order to reduce the bullwhip effect if the lead time is longer.

The impacts of demand substitution on the lower bounds of the bullwhip effect of the two products are correlated with the values of the autoregressive coefficients ρ_1 and ρ_2 as well as the parity of p. For product 1, i.e., the product being used to substitute another one, generally, the demand substitution will decrease the lower bound of the bullwhip effect of product 1 except for odd values of p and negative ρ_1. When the successive period demands are negatively related and the number of observation used in MA is odd, the demand substitution will increase the lower bound of the bullwhip effect of product 1.

For product 2, when $\rho_1 > 0$, $\rho_2 > 0$, the lower bound of the bullwhip effect of product 2 is a decreasing function of λ. When $\rho_1 < 0$ or $\rho_2 < 0$, the behavior is hard to predict. We prove that when $\frac{\rho_1}{1-\lambda} + \rho_2 < 0$, for even values of p, the lower bound of the bullwhip effect of product 2 will be larger with demand substitution.

There are a few ways to extend the research in this paper. First, the findings of this research are expected to serve as a starting point to explore the impact of demand substitution on the bullwhip effect in more complex systems, including multi-stage supply chain network, different inventory policy, and complex forecasting method of lead-time demand estimation and different demand processes. Secondly, the impact of the two product substitution to the bullwhip effect can be extended by modeling N products under "one-way substitution" and/or "two-way substitution" scenarios. Thirdly, we assumed a deterministic proportion of substitution. However, literatures from marketing and psychological research suggest that the customer purchase pattern cannot be deterministic. The purchase decision is largely influenced by the surrounding environment, social status, emotional condition, and other subjective factors. The research of this paper can be extended if the demand substitution model could capture the customer behavior more accurately.

Appendix

The derivation process of $E(D_{t,i})$ and $\mathrm{Var}(D_{t,i})$

When the autoregressive demand process is stationary, we have $E(D_{t,i}) = E(D_{t-1,i}) = E(D_{t-2,i}) = \cdots = E(D_i)$ and $\mathrm{Var}(D_{t,i}) = \mathrm{Var}(D_{t-1,i}) = \mathrm{Var}(D_{t-2,i}) = \cdots = \mathrm{Var}(D_i)$

$$D_{t,1} = \mu_1 + \rho_1 D_{t-1,1} + \epsilon_{t,1} + \lambda D_{t,1}$$
$$(1-\lambda)D_{t,1} = \mu_1 + \rho_1 D_{t-1,1} + \epsilon_{t,1}$$

$$(1-\lambda)E(D_{t,1}) = E(\mu_1) + \rho_1 E(D_{t-1,1}) + E(\epsilon_{t,1})$$
$$(1-\lambda)E(D_1) = \mu_1 + \rho_1 E(D_1) + 0$$
$$E(D_1) = \frac{\mu_1}{1-\rho_1-\lambda}$$

$$(1-\lambda)^2\mathrm{Var}(D_{t,1}) = \mathrm{Var}(\mu_1) + {\rho_1}^2\mathrm{Var}(D_{t-1,1}) + \mathrm{Var}(\epsilon_{t,1})$$
$$(1-\lambda)^2\mathrm{Var}(D_1) = 0 + {\rho_1}^2\mathrm{Var}(D_1) + \sigma_1^2$$
$$\mathrm{Var}(D_1) = \frac{\sigma_1^2}{(1-\lambda)^2 - \rho_1^2}$$

$$D_{t,2} = \mu_2 + \rho_2 D_{t-1,2} + \epsilon_{t,2} - \lambda D_{t,1}$$

$$E(D_{t,2}) = E(\mu_2) + \rho_2 E(D_{t-1,2}) + E(\epsilon_{t,2}) - \lambda E(D_{t,1})$$
$$E(D_2) = \mu_2 + \rho_2 E(D_2) + 0 - \lambda E(D_1)$$
$$(1-\rho_2)E(D_2) = \mu_2 - \lambda \frac{\mu_1}{1-\rho_1-\lambda}$$
$$E(D_{t,2}) = \frac{\mu_2(1-\rho_1-\lambda) - \lambda\mu_1}{(1-\rho_1-\lambda)(1-\rho_2)}$$

$$\mathrm{Var}(D_{t,2}) = \mathrm{Var}(\mu_2) + {\rho_2}^2 \mathrm{Var}(D_{t-1,2}) + \mathrm{Var}(\epsilon_{t,2}) - \lambda^2 \mathrm{Var}(D_{t,1})$$
$$\mathrm{Var}(D_2) = 0 + {\rho_2}^2 var(D_2) + {\sigma_2}^2 - \lambda^2 \mathrm{Var}(D_1)$$
$$(1-\rho_2^2)\mathrm{Var}(D_2) = {\sigma_2}^2 - \lambda^2 \frac{\sigma_1^2}{(1-\lambda)^2 - \rho_1^2}$$
$$\mathrm{Var}(D_{t,2}) = \frac{[(1-\lambda)^2 - \rho_1^2]\sigma_2^2 - \lambda^2\sigma_1^2}{[(1-\lambda)^2 - \rho_1^2](1-\rho_2^2)}$$

The derivation process of the further expression of $q_{t,1}$

$$\begin{aligned} q_{t,1} &= y_{t,1} - y_{t-1,1} + D_{t-1,1} \\ &= (\hat{D}^L_{t,1} + z_1\hat{\sigma}^L_{t,1}) - (\hat{D}^L_{t-1,1} + z_1\hat{\sigma}^L_{t-1,1}) + D_{t-1,1} \\ &= (\hat{D}^L_{t,1} - \hat{D}^L_{t-1,1}) + D_{t-1,1} + z_1(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \\ &= \frac{L}{p}\left(\sum_{i=1}^{p} D_{t-i,1} - \sum_{i=1}^{p} D_{t-1-i,1}\right) + D_{t-1,1} \\ &\quad + z_1(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \\ &= \frac{L}{p}(D_{t-1,1} - D_{t-p-1,1}) + D_{t-1,1} + z_1(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \\ &= (1 + L/p)D_{t-1,1} - (L/p)D_{t-p-1,1} + z_1(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \end{aligned}$$

The derivation process of the expression of $\mathrm{Var}(q_{t,1})$

$$\begin{aligned} \mathrm{Var}(q_{t,1}) &= \mathrm{Var}[(1+L/p)D_{t-1,1} - (L/p)D_{t-p-1,1} \\ &\quad + z_1(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1})] \\ &= (1+L/p)^2 Var(D_{t-1,1}) - 2(L/p)(1+L/p) \\ &\quad \times \mathrm{Cov}(D_{t-1,1}, D_{t-p-1,1}) \\ &\quad + (L/p)^2 Var(D_{t-p-1,1}) + z_1^2 Var(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \\ &\quad + 2z_1(1+2L/p) \times \mathrm{Cov}(D_{t-1,1}, \hat{\sigma}^L_{t,1}) \\ &= \left(1 + 2\frac{L}{p} + 2\frac{L^2}{p^2}\right)\mathrm{Var}(D_1) - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right) \\ &\quad \mathrm{Cov}(D_{t-1,1}, D_{t-p-1,1}) \quad + z_1^2 Var(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \\ &\quad + 2z_1(1+2L/p)\mathrm{Cov}(D_{t-1,1}, \hat{\sigma}^L_{t,1}) \end{aligned} \tag{13}$$

To further simplify Equation 9, we need to calculate $\mathrm{Cov}(D_{t\text{-}1,1}, D_{t\text{-}p\text{-}1,1})$ and $Cov(D_{t-1,1}, \hat{\sigma}^L_{t,1})$

$$\begin{aligned} &\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,1}) \\ &= \mathrm{Cov}\left(\frac{1}{1-\lambda}(\mu_1 + \rho_1 D_{t-2,1} + \epsilon_{t,1}), D_{t-p-1,1}\right) \\ &= \underbrace{\mathrm{Cov}\left(\frac{\mu_1}{1-\lambda}, D_{t-p-1,1}\right)}_{=0} + \frac{\rho_1}{1-\lambda}\mathrm{Cov}(D_{t-2,1}, D_{t-p-1,1}) \\ &\quad + \frac{1}{1-\lambda}\underbrace{\mathrm{Cov}(\epsilon_{t,1}, D_{t-p-1,1})}_{=0} \\ &= \frac{\rho_1}{1-\lambda}\mathrm{Cov}(D_{t-2,1}, D_{t-p-1,1}) \\ &\vdots \\ &= \left(\frac{\rho_1^p}{(1-\lambda)^p}\right)\mathrm{Cov}(D_{t-p-1,1}, D_{t-p-1,1}) \\ &= \frac{\rho_1^p}{(1-\lambda)^p}\mathrm{Var}(D_1) \end{aligned} \tag{14}$$

Note that $Cov(\frac{\mu_1}{1-\lambda}, D_{t-p-1,1}) = 0$, because $\frac{\mu_1}{1-\lambda}$ is a constant ($\mathrm{Cov}(X,a) = 0$). $Cov(\epsilon_{t,1}, D_{t-p-1,1}) = 0$, because $\epsilon_{t,1}, D_{t-p-1,1}$ are independent from each other.

Ryan [20] proved the following result that can further simplify Equation 9. Assume the customer demands seen by a retailer are random variables of the form as $D_t = \mu + \rho D_{t-1} + \epsilon_t$ and the error terms ϵ_t are i.i.d. from a symmetric distribution with mean 0 and variance σ^2. Let the estimate of the standard deviation of forecast error of lead-time demand be $\hat{\sigma}^L_t = C_{L,\rho}\sqrt{\frac{\sum_{i=j}^{p}(D_{t-j} - \hat{D}_{t-j})^2}{p}}$, then

$$\mathrm{Cov}(D_{t-j}, \hat{\sigma}^L_t) = 0, \forall i = 1, \ldots, p. \tag{15}$$

By applying the results of Equation 28 and Equation 29, the expression about $\mathrm{Var}(q_{t,1})$ can be further simplified.

$$\begin{aligned} \mathrm{Var}(q_{t,1}) &= \left(1 + 2\frac{L}{p} + 2\frac{L^2}{p^2}\right)\mathrm{Var}(D_1) - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right) \\ &\quad \mathrm{Cov}(D_{t-1,1}, D_{t-p-1,1}) + z_1^2 Var(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \\ &\quad + 2z_1(1+2L/p)\mathrm{Cov}(D_{t-1,1}, \hat{\sigma}^L_{t,1}) \\ &= \left(1 + 2\frac{L}{p} + 2\frac{L^2}{p^2}\right)\mathrm{Var}(D_1) \\ &\quad - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\frac{\rho_1^p}{(1-\lambda)^p}\mathrm{Var}(D_1) \\ &\quad + z_1^2 Var(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) + 0 \\ &= \mathrm{Var}(D_1)\left[1 + \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\left(1 - \frac{\rho_1^p}{(1-\lambda)^p}\right)\right] \\ &\quad + z_1^2 Var(\hat{\sigma}^L_{t,1} - \hat{\sigma}^L_{t-1,1}) \end{aligned} \tag{16}$$

The derivation process of the further expression of $q_{t,2}$

$$
\begin{aligned}
q_{t,2} &= y_{t,2} - y_{t-1,2} + D_{t-1,2} \\
&= (\hat{D}^L_{t,2} + z_1\hat{\sigma}^L_{t,2}) - (\hat{D}^L_{t-1,2} + z_1\hat{\sigma}^L_{t-1,2}) + D_{t-1,2} \\
&= (\hat{D}^L_{t,2} - \hat{D}^L_{t-1,2}) + D_{t-1,2} + z_2(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}) \\
&= \frac{L}{p}\left(\sum_{i=1}^{p} D_{t-i,2} - \sum_{i=1}^{p} D_{t-1-i,2}\right) + D_{t-1,2} \\
&\quad + z_2(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}) \\
&= \frac{L}{p}(D_{t-1,2} - D_{t-p-1,2}) + D_{t-1,2} + z_2(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}) \\
&= (1 + L/p)D_{t-1,2} - (L/p)D_{t-p-1,2} + z_2(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2})
\end{aligned}
$$

The derivation process of the expression of $\mathrm{Var}(q_{t,2})$

$$
\begin{aligned}
\mathrm{Var}(q_{t,2}) &= \mathrm{Var}[(1 + L/p)D_{t-1,2} - (L/p)D_{t-p-1,2} \\
&\quad + z_2(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2})] \\
&= (1 + L/p)^2 Var(D_{t-1,2}) - 2(L/p)(1 + L/p) \\
&\quad \times \mathrm{Cov}(D_{t-1,2}, D_{t-p-1,2}) \\
&\quad + (L/p)^2 Var(D_{t-p-1,2}) + z_2^2 Var(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}) \\
&\quad + 2z_2(1 + 2L/p) \times \mathrm{Cov}(D_{t-1,2}, \hat{\sigma}^L_{t,2}) \\
&= \left(1 + 2\frac{L}{p} + 2\frac{L^2}{p^2}\right)\mathrm{Var}(D_2) - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right) \\
&\quad \mathrm{Cov}(D_{t-1,2}, D_{t-p-1,2}) \\
&\quad + z_2^2 Var(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}) \\
&\quad + \underbrace{2z_2(1 + 2L/p)\mathrm{Cov}(D_{t-1,2}, \hat{\sigma}^L_{t,2})}_{=0}
\end{aligned} \tag{17}
$$

Now we calculate $\mathrm{Cov}(D_{t-1,2}, D_{t-p-1,2})$

$$
\begin{aligned}
\mathrm{Cov}(D_{t-1,2}, D_{t-p-1,2}) &= \mathrm{Cov}(\mu_2 + \rho_2 D_{t-2,2} \\
&\quad + \epsilon_{t-1,2} - \lambda D_{t-1,1}, D_{t-p-1,2}) \\
&= \underbrace{\mathrm{Cov}(\mu_2, D_{t-p-1,2})}_{=0} + \rho_2 Cov(D_{t-2,2}, D_{t-p-1,2}) \\
&\quad + \underbrace{\mathrm{Cov}(\epsilon_{t-1,2}, D_{t-p-1,2})}_{=0} - \lambda\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2}) \\
&= \rho_2 Cov(D_{t-2,2}, D_{t-p-1,2}) - \lambda\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2}) \\
&\vdots \\
&= \rho_2^p Var(D_2) - \lambda\sum_{i=0}^{p-1}\rho_2^i Cov(D_{t-1-i,1}, D_{t-p-1,2})
\end{aligned}
$$

We assume that the covariance is only affected by the number of periods which are taken into consideration. That is

$$
\begin{aligned}
&\mathrm{Cov}(D_{t,1}, D_{t-p,2}) = \mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2}) \\
&D_{t,1} = \frac{1}{1-\lambda}(\mu_1 + \rho_1 D_{t-1,1} + \epsilon_{t,1}) \\
&D_{t-p,2} = \mu_2 + \rho_2 D_{t-p-1,2} + \epsilon_{t-p,2} - \lambda D_{t-p,1} \\
&\mathrm{Cov}(D_{t,1}, D_{t-p,2}) = \mathrm{Cov}(\frac{1}{1-\lambda}(\mu_1 + \rho_1 D_{t-1,1} + \epsilon_{t,1}), \\
&\quad \mu_2 + \rho_2 D_{t-p-1,2} + \epsilon_{t-p,2} - \lambda D_{t-p,1}) \\
&\quad = \frac{\rho_1\rho_2}{1-\lambda}\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2}) - \frac{\lambda\rho_1}{1-\lambda}\mathrm{Cov}(D_{t-1,1}, D_{t-p,1}) \\
&\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2}) = \frac{\rho_1\rho_2}{1-\lambda}\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2}) \\
&\quad - \frac{\lambda\rho_1}{1-\lambda}\mathrm{Cov}(D_{t-1,1}, D_{t-p,1}) \\
&\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2}) = \frac{-\lambda(1-\lambda)\rho_1^p}{(1-\lambda-\rho_1\rho_2)(1-\lambda)^p}\mathrm{Var}(D_1) \\
&\mathrm{Cov}(D_{t-2,1}, D_{t-p-1,2}) = \frac{-\lambda(1-\lambda)\rho_1^{p-1}}{(1-\lambda-\rho_1\rho_2)(1-\lambda)^{p-1}}\mathrm{Var}(D_1) \\
&\vdots \\
&\mathrm{Cov}(D_{t-i,1}, D_{t-p-1,2}) = \frac{-\lambda(1-\lambda)\rho_1^{p-i+1}}{(1-\lambda-\rho_1\rho_2)(1-\lambda)^{p-i+1}}\mathrm{Var}(D_1) \\
&\vdots \\
&\mathrm{Cov}(D_{t-i,1}, D_{t-p-1,2}) = \frac{-\lambda(1-\lambda)\rho_1^2}{(1-\lambda-\rho_1\rho_2)(1-\lambda)^2}\mathrm{Var}(D_1)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathrm{Cov}(D_{t-1,2}, D_{t-p-1,2}) \\
&\quad = \rho_2^p Var(D_2) - \lambda\mathrm{Cov}(D_{t-1,1}, D_{t-p-1,2})\sum_{i=0}^{p}\rho_2^i\left(\frac{1-\lambda}{\rho_1}\right)^i \\
&\quad = \rho_2^p Var(D_2) - \lambda\sum_{i=0}^{p}\rho_2^i Cov(D_{t-1-i,1}, D_{t-p-1,2}) \\
&\quad = \rho_2^p Var(D_2) + \frac{\lambda^2(1-\lambda)\mathrm{Var}(D_1)}{(1-\lambda-\rho_1\rho_2)}\sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right)
\end{aligned}
$$

Substitute the above equation to Eq. 17, we have

$$
\begin{aligned}
\mathrm{Var}(q_{t,2}) &= \left(1 + 2\frac{L}{p} + 2\frac{L^2}{p^2}\right)\mathrm{Var}(D_2) - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right) \\
&\quad \mathrm{Cov}(D_{t-1,2}, D_{t-p-1,2}) + z_2^2 Var(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2}) \\
&= \left(1 + 2\frac{L}{p} + 2\frac{L^2}{p^2}\right)\mathrm{Var}(D_2) \\
&\quad - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\rho_2^p Var(D_2) \\
&\quad - \left(\frac{2L}{p} + \frac{2L^2}{p^2}\right)\frac{\lambda^2(1-\lambda)\mathrm{Var}(D_1)}{(1-\lambda-\rho_1\rho_2)}\sum_{i=0}^{p-1}\left(\frac{\rho_2^i\rho_1^{p-i+1}}{(1-\lambda)^{p-i+1}}\right) \\
&\quad + z_2^2 Var(\hat{\sigma}^L_{t,2} - \hat{\sigma}^L_{t-1,2})
\end{aligned} \tag{18}
$$

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