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ORIGINAL PAPER

Modeling and stability analysis of autonomously controlled production networks

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Abstract We present methods and tools for modeling autonomously controlled production networks and investigation of their stability properties. Production networks are described as interconnected dynamical systems of two types: systems of ordinary differential equations and time-delay systems. In particular with the help of time-delays, we incorporate transportation times and implement an autonomous control method, namely the queue length estimator. By stability, we mean roughly speaking, boundedness of the state of a system (e.g., the inventory level or the work in progress) over the time under bounded external inputs. In our stability analysis, we consider the case, when all the subsystems describing logistics locations are stable. We derive sufficient conditions that guarantee stability of the network. To this end, we utilize Lyapunov functions and a small gain condition.

Keywords Production networks · Modeling · Stability analysis · Lyapunov functions

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1 Introduction

Production, supply networks, and other logistic structures are typical examples of complex systems with a nonlinear and sometimes chaotic dynamics. Their dynamics is subject to different perturbations due to changes on market, changes in customer behavior, information and transport congestions, unreliable elements of the network etc. One of the approaches to handle such complex systems is to shift from centralized to decentralized or autonomous control, i.e., to allow the entities of a network to make their own decisions based on some given rules and available local information [29, 30]. However, a system emerging in this way can become unstable and hence be not effective according to the logistic performance.

Typical examples of unstable behavior are unbounded growth of unsatisfied orders or unbounded growth of the queue of the workload to be processed by a machine. This causes a loss of customers and high inventory costs, respectively. To avoid instability of a network, one needs to investigate its behavior in advance. Mathematical modeling and analysis provide helpful tools for design, optimization, and control of such networks and for deeper understanding of their dynamical properties.

1.1 Production networks

The term production network is used to describe company or cross-company-owned networks with geographically dispersed plants. The primary objective of production networks is to achieve economies of scale through joint planning of production processes, a mutual use of common resources, and integrated planning of value added processes [28]. These types of networks can react quickly to perturbations due to redundancies of common resources.

But high flexibility causes interdependencies between production processes in different plants, e.g., allocation problems for products or planning of transports and transport capacity [1, 18]. Therefore, production planning and control (PPC) of production networks has to cover these tasks and also has to provide methods for an integrated planning and synchronization within the network, including planning of sales and inventory [28]. Under highly dynamic and complex conditions current PPC methods cannot cope with disturbances or unforeseen events in an appropriate manner [15]. This can cause uncertainties of lead times or unsteadiness of schedules, and it can also lead to instability.

1.2 Autonomous control

The main idea of autonomous cooperating logistic processes is to enable intelligent logistic objects to route themselves through a logistic network according to their own objectives and to make and execute decisions, based on local information [29, 30]. In this context, intelligent logistic objects can be physical or material objects (e.g., parts or machines) as well as immaterial objects (e.g., production orders, information). It has already been shown that different autonomous control methods can help to increase the logistics performance and robustness of single production systems [19, 21]. Due to the high structural and dynamical complexity of production networks, one can expect that autonomous control has a positive effect on the dynamical behavior of these networks. This was confirmed by investigations of the performance of autonomously controlled production networks [20]. On the other hand, autonomously controlled production networks can show a sudden change of the dynamical systems behavior in dependence of varying start parameters and the logistics performance collapses in the sense of unpredictable and increasing throughput times and growing inventory [22]. Thus, investigations of autonomously controlled production networks stability are essential to identify such turning points of dynamical systems behavior.

The autonomous control to be modeled and used in this contribution is based on the queue length estimator (QLE), which was investigated in previous papers together with other existing autonomous control methods [2, 21, 23]. The QLE enables parts to choose the next transportation way to an entity of the network according to the local information about their current amount of the queuing workload.

1.3 Mathematical modeling and stability analysis

Roughly speaking, for production networks stability means that the state of the network remains bounded over time under bounded external inputs.

The state of the system is the set of variables that determines the evolution of the system (if the external inputs are given). In this contribution, we will consider the state of the system as the number of unprocessed parts, which is the sum of the queue length and the work in progress (WIP).

Thus, stable behavior of the network is decisive for the performance and vitality of a network. To design stable logistic networks, we are going to apply tools from mathematical systems theory. In this context, mathematical models describing network's behavior are needed.

For manufacturing systems, parameters assuring stable behavior can be found by using different models: fluid models [4], re-entrant lines [5], or manufacturing systems with different job types [6]. An approach with flows of multiple fluids was used to analyze the stability region of an autonomously controlled shop floor scenario [24]. Scholz-Reiter et al. [25] presented a fluid model of a production network and obtained a stability region for a scenario with two locations and three types of products. First approaches have already been done to derive stability conditions of autonomously controlled production networks [7].

In this contribution, a production network is described as an interconnection of many dynamical subsystems that model logistic locations. To cope with different dynamical characteristics of the network, we develop two types of models: systems, based on ordinary differential equations (ODEs) and time-delay systems. Time-delay systems are described by functional differential equations and take transportation times into account in contrast to models, based on ODEs. Both models describe continuous material flows in the production network. The QLE is implemented in both types of models.

We study input-to-state stability [26] of production networks. Our stability analysis is based on the Lyapunov function theory and small-gain theorems. We divide the analysis in several steps: at the first step, we describe the network's behavior by a mathematical model according to the type of its dynamics. Then, we are looking for Lyapunov functions and the corresponding Lyapunov gains to establish stability of each subsystem. If all subsystems are stable, then we apply the so-called small-gain condition that takes into account the interconnection structure of the network. If this condition is satisfied, then the stability of the network is proved; otherwise, we cannot conclude whether the network is stable or not. But we can repeat the stability analysis choosing another Lyapunov function candidate and/or gains. Note that the choices of a Lyapunov function candidate and the gains are rather heuristic. This framework is described in Fig. 1. Note that we provide only sufficient conditions for stability of a network.

This procedure can be applied to general nonlinear large-scale systems to perform a stability analysis and to derive bounds for parameters of a logistic system for which its

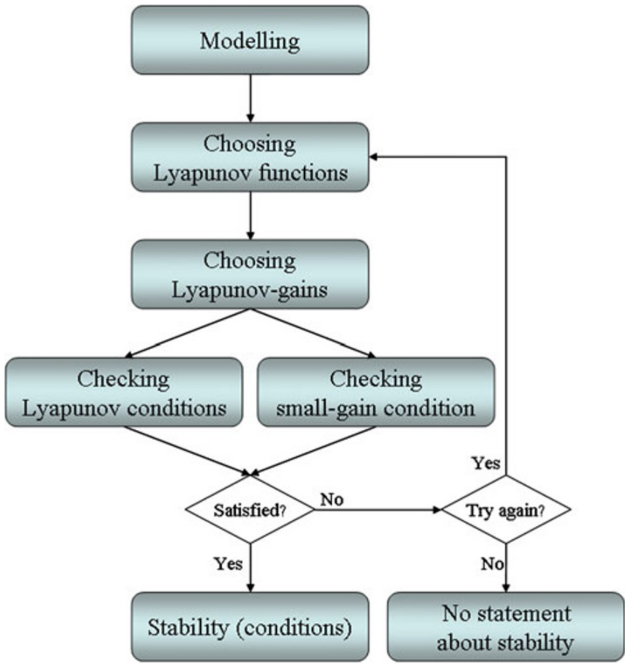


Fig. 1 Scheme of the stability analysis procedure

behavior is stable. These bounds can be used during the design and management of production networks to achieve stable behavior.

The structure of the contribution is as follows. In Sect. 2, we give the necessary notions of the dynamical systems and review the stability results for them, namely ODE systems are considered in Sect. 2.1 and time-delay systems in Sect. 2.2. These results will be used in Sect. 3 for modeling and analyzing the behavior of logistics networks with and without time-delays. The application will be supplemented by numerical simulations in Matlab for a certain scenario of a production network in Sect. 4. Section 5 concludes the contribution and outlines some approaches for the future work.

2 Modeling methods and mathematical stability theory

In this section, we introduce two different methods to model dynamical networks, such as production networks. Furthermore, the stability theory for these methods is presented.

2.1 Ordinary differential equations

One possibility to model production networks is ordinary differential equations (ODEs), see for example [14]. ODEs describe the evolution of the state of the system with continuous time $t \in \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$.

By x^T we denote the transposition of a vector $x \in \mathbb{R}^n$, $n \in \mathbb{N}$. $\mathbb{R}^n_+$ denotes the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$ where we use the standard partial order for $x, y \in \mathbb{R}^n$ given by

$$x \geq y \Leftrightarrow x_i \geq y_i, \quad i = 1, \ldots, n \quad \text{and} \quad x \not\geq y \Leftrightarrow \exists i : x_i < y_i.$$

To handle the external inputs of the system, we use ODE with inputs of the form

$$\dot{x}(t) = f(x(t), u(t)), \ t \in \mathbb{R}_+, \tag{1}$$

where $x(t) \in \mathbb{R}^N$ denotes the state of the system at time t, u is the essentially bounded measurable external input, i.e., $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^M)$ and $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ describes the system dynamics. The norm in the space $L_\infty(\mathbb{R}_+, \mathbb{R}^M)$ is given by $\|u\|_\infty := \text{ess sup}_{t \in [0,\infty)} |u(t)|$, where $|\cdot|$ denotes the Euclidean norm.

To have existence and uniqueness of a solution of a system of the form (1), the function f is assumed to be a locally Lipschitz continuous function. The solution is denoted by $x\,(t; x_0, u)$ or $x\,(t)$ for short, where x_0 is the initial condition at time $t = 0$.

In general, production networks consist of $n \in \mathbb{N}$ interconnected systems of the form

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u_i(t)), \ t \in \mathbb{R}_+, \ i = 1, \ldots, n, \tag{2}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$ and $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + M_i} \to \mathbb{R}^{N_i}$ are locally Lipschitz continuous functions. Here, x_j, $j \neq i$ can be interpreted as internal inputs of the i-th subsystem, and the solution is denoted by $x_i\,(t; x_i^0, x_j, j \neq i, u_i)$ or $x_i(t)$ for short, where x_i^0 is the initial condition at time $t = 0$.

If we define $N := \sum_{i=1}^n N_i$, $M := \sum_{i=1}^n M_i$, $x := (x_1^T, \ldots, x_n^T)^T$, $u := (u_1^T, \ldots, u_n^T)^T$ and $f = (f_1^T, \ldots, f_n^T)^T$, then the interconnected system of the form (2) can be written as one single system of the form (1), which we call the whole system.

The purpose of this paper is to analyze production networks, which can be written in the form (2), in view of stability.

Definition 1 For the stability analysis, the following classes of functions are useful:

$$\mathcal{P} := \{f : \mathbb{R}^n \to \mathbb{R}_+ |\ f(0) = 0,\ f(x) > 0,\ x \neq 0\},$$

$$\mathcal{K} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ |\ \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and strictly increasing}\},$$

$$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} |\ \gamma \text{ is unbounded}\},$$

$$\mathcal{L} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ |\ \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\},$$

$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ |\ \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}.$$

We will call functions of class $\mathcal{P}$ *positive definite*.

In the rest of the paper, by $x, y > 0$, we mean that $x > 0$ and $y > 0$ holds. Now, we introduce the following stability notion:

Definition 2

1. System (1) is called *locally input-to-state stable (LISS)* if there exist constants $\rho,\ \rho_u > 0$ and functions $\beta \in \mathcal{KL},\ \gamma \in \mathcal{K}$ such that for all initial values $|x_0| \le \rho$ and all inputs $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^M)$: $\|u\|_\infty \le \rho_u$ the inequality
$$|x(t)| \le \max\{\beta(|x_0|, t), \gamma(\|u\|_\infty)\}$$
is satisfied for all $t \in \mathbb{R}_+$. Function γ is called (nonlinear) gain.
2. The i-th subsystem of (2) is called *LISS* if there exist constants $\rho_i,\ \rho_{ij},\ \rho_i^u > 0$ and functions $\gamma_{ij},\ \gamma_i \in \mathcal{K}$ and $\beta_i \in \mathcal{KL}$ such that for all initial values $|x_i^0| \le \rho_i$ and all inputs $\|x_j\|_\infty \le \rho_{ij},\ \|u_i\|_\infty \le \rho_i^u$ the inequality
$$|x_i(t)| \le \max\left\{\beta_i(|x_i^0|, t), \max_{j\neq i} \gamma_{ij}\left(\|x_j\|_\infty\right), \gamma_i(\|u_i\|_\infty)\right\}$$

is satisfied for all $t \in \mathbb{R}_+$. γ_{ij} and γ_i are called (nonlinear) gains.

Note that, if $\rho, \rho_u = \infty$, then the system (1) is called *(globally) ISS* and if $\rho_i, \rho_{ij}, \rho_i^u = \infty$, then the i-th subsystem of (2) is called (globally) ISS.

In particular, LISS (for $|x_i^0| \le \rho_i, \|x_j\|_\infty \le \rho_{ij}, \|u_i\|_\infty \le \rho_i^u$) and ISS (for all initial values and external and internal inputs) guarantee that the norm of the trajectories of each subsystem is bounded.

An important tool to verify LISS and ISS, respectively, of a system of the form (2) are Lyapunov functions.

Definition 3 We assume that for each subsystem of the interconnected system (1) there exists a function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous and positive definite. Then, for $i = 1, \ldots, n$ the function V_i is called a *LISS Lyapunov function of the* $i - th$ *subsystem of* (2) if V_i satisfies the following two conditions: There exist functions $\psi_{1i},\ \psi_{2i} \in \mathcal{K}_\infty$ such that

$$\psi_{1i}(|x_i|) \le V_i(x_i) \le \psi_{2i}(|x_i|),\ \forall\, x_i \in \mathbb{R}^{N_i} \tag{3}$$

and there exist $\gamma_{ij},\ \gamma_i \in \mathcal{K}$, a positive definite function μ_i and constants $\rho_i,\ \rho_{ij},\ \rho_i^u > 0$ such that

$$V_i(x_i) \ge \max\left\{\max_{j\neq i} \gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\right\}$$
$$\Rightarrow\ \nabla V_i(x_i) \cdot f_i(x, u) \le -\mu_i(V_i(x_i)) \tag{4}$$

for almost all $x_i \in \mathbb{R}^{N_i},\ |x_i^0| \le \rho_i,\ |x_j| \le \rho_{ij},\ u_i \in \mathbb{R}^{M_i},\ |u_i| \le \rho_i^u,\ \gamma_{ii} = 0$, where ∇ denotes the gradient of the function V_i. Functions γ_{ij} are called *LISS Lyapunov gains*.

Note that, if $\rho_i, \rho_{ij}, \rho_i^u = \infty$, then the LISS Lyapunov function of the i-th subsystem becomes an ISS Lyapunov function of the i-th subsystem (see [12]). In general, the LISS Lyapunov gains differ from the gains in Definition 2.

The condition (3) means that V_i is positive definite and radially unbounded. Function V_i can be interpreted as the "energy" of the system. The condition (4) means that outside of the region $\{x_i:\ V_i(x_i) < \max\{\max_{j\neq i} \gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\}\}$ the "energy" of the system is decreasing. In particular, for every given external and internal inputs with finite norms, the energy of the system is bounded, which implies, by (3), that also the trajectory of the i-th subsystem remains bounded for all time $t > 0$.

Furthermore, we define the *gain-matrix* $\Gamma := (\gamma_{ij})_{n\times n}$, $i, j = 1, \ldots, n,\ \gamma_{ii} = 0$, which defines a map $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma(s) := \left(\max_j \gamma_{1j}(s_j), \ldots, \max_j \gamma_{nj}(s_j)\right)^T,\ s \in \mathbb{R}_+^n. \tag{5}$$

Note that the matrix Γ describes in particular the interconnection structure of the network; moreover, it contains the information about the mutual influence between the subsystems, which can be used to verify the (L)ISS property of networks.

Definition 4 Γ satisfies the *local small gain condition (LSGC) on* $[0, w^*]$, provided that

$$\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\ge s,\ \forall s \in [0, w^*],\ s \neq 0. \tag{6}$$

Notation $\not\ge$ denotes that there is at least one component $i \in \{1, \ldots, n\}$ such that $\Gamma(s)_i < s_i$.

To check whether the interconnected system of the form (1) possesses the LISS property, we use the scheme in Fig. 1. To this end, one has to find a LISS Lyapunov function for each subsystem. If there exists a LISS Lyapunov function for each subsystem, then this subsystem possesses the LISS property. Furthermore, if the LISS Lyapunov gains satisfy the local small-gain condition, then the whole system of the form (1) is LISS, which we recall in the following theorem (see [11]):

Theorem 1 *Consider the interconnected system* (2)*, where each subsystem has an LISS Lyapunov function* V_i*. If the corresponding gain-matrix* Γ *satisfies the local small-gain condition* (6)*, then there exist constants* $\rho,\ \rho_u > 0$ *such that the whole system of the form* (1) *is LISS.*

In [10], a similar ISS small-gain theorem for general networks was proved, where the small-gain condition is of the form

$$\Gamma(s) \not\ge s,\ \forall\, s \in \mathbb{R}_+^n \backslash \{0\}.$$

2.2 Time-delay systems

In this section, we introduce systems with time-delays that allow modeling of transportation times in logistic networks: material leaves one production location at time t and reaches the following location at time $t+\theta$, where $\theta > 0$ is the transportation time between these two production locations. Time-delay systems are described by differential equations of the form

$$\dot{x}(t) = f(x^t, u). \tag{7}$$

Here, the term $x^t : \tau \mapsto x(t+\tau), \tau \in [-\theta, 0], x^t \in C([-\theta,0]; \mathbb{R}^N)$ represents the state of the system at time t, where $C([-\theta,0];\mathbb{R}^N)$ denotes the space of continuous functions defined on $[-\theta,\ 0]$ equipped with the norm $\|x^t\|_{[-\theta,0]} := \sup_{\tau\in[-\theta,0]}|x(t+\tau)|$ and values in $\mathbb{R}^N$. In (7), u is again an external input, $f : C([-\theta,0];\mathbb{R}^N) \times \mathbb{R}^M \to \mathbb{R}^N$ describes the dynamics of the system that is now dependent also on the previous values of the function x.

In other words, the state of the time-delay system at the time t is the set of values of the function x in the period $[t-\theta,\ t]$, and θ can be interpreted as the maximal involved delay. We assume that the conditions for the existence and uniqueness of a solution of (7) are satisfied. Let the initial state be given by the function $\xi \in C([-\theta,0];\mathbb{R}^N)$.

The stability notions introduced in the previous section can be defined for time-delay systems as well:

Definition 5 System (7) is called LISS if there exist constants $\rho,\ \rho_u > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for every initial condition $\|\xi\|_{[-\theta,0]} \le \rho$, every external input $\|u\|_\infty \le \rho_u$ and for all $t \in \mathbb{R}_+$, it holds that

$$|x(t)| \le \max\left\{\beta\left(\|\xi\|_{[-\theta,0]}, t\right), \gamma\left(\|u\|_\infty\right)\right\}, \tag{8}$$

where $\xi \in C([-\theta,0],\mathbb{R}^N)$.

Remark 1 An equivalent definition of LISS of time-delay systems can be obtained by replacing the inequality (8) in Definition 2 by the inequality

$$\|x^t\|_{[-\theta,0]} \le \max\left\{\beta\left(\|\xi\|_{[-\theta,0]}, t\right), \gamma\left(\|u\|_\infty\right)\right\}. \tag{9}$$

Really, if the system (7) is LISS in the form (9), then it is LISS according to Definition 2, because of $|x(t)| \le \|x^t\|_{[-\theta,0]}$.

In the other direction, if (7) is LISS according to the Definition 2, then there exist $\rho, \rho_u > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for every initial condition $\|\xi\|_{[-\theta,0]} \le \rho$, every external input $\|u\|_\infty \le \rho_u$ and for all $t > \theta$ it holds

$$\begin{aligned}\|x^t\|_{[-\theta,0]} &= \sup_{\tau\in[-\theta,0]} |x(t+\tau)| \\ &\le \max\left\{\sup_{\tau\in[-\theta,0]} \beta\left(\|\xi\|_{[-\theta,0]}, t+\tau\right), \gamma\left(\|u\|_\infty\right)\right\} \\ &= \max\left\{\beta\left(\|\xi\|_{[-\theta,0]}, t-\theta\right), \gamma\left(\|u\|_\infty\right)\right\}.\end{aligned}$$

For $t \in [0,\theta]$ it holds $\|x^t\|_{[-\theta,0]} \le \max\left\{\beta\left(\|\xi\|_{[-\theta,0]}, 0\right), \gamma\left(\|u\|_\infty\right)\right\}$. Now define for all $r \ge 0$, $t \ge 0$ the function

$$\tilde{\beta}(r,t) = \begin{cases} \beta(r, t-\theta), & t > \theta \\ (\theta - t) + \beta(r, 0), & t \in [0,\theta]. \end{cases}$$

One can simply check that $\tilde{\beta} \in \mathcal{KL}$. Now, for every initial condition $\|\xi\|_{[-\theta,0]} \le \rho$, every external input $\|u\|_\infty \le \rho_u$ and for all $t > 0$ it holds

$$\|x^t\|_{[-\theta,0]} \le \max\left\{\tilde{\beta}\left(\|\xi\|_{[-\theta,0]}, t\right), \gamma\left(\|u\|_\infty\right)\right\},$$

therefore the system (7) is LISS also in the form (9).

If we consider n interconnected systems, then we write each subsystem as

$$\dot{x}_i(t) = f_i\left(x_1^t, \ldots, x_n^t, u_i(t)\right), \tag{10}$$

where $x_j^t := x_j(t+\tau)$, $\tau \in [-\theta, 0]$ can be interpreted as internal input of the i-th subsystem, $i = 1,\ldots,n$. The initial functions are given by $\xi_i \in C([-\theta,0];\mathbb{R}^{N_i})$. Again, this network can be written in the form (7). The notion of LISS for interconnected time-delay systems is as follows:

Definition 6 The i-th subsystem of (10) is called LISS if there exist constants $\rho_i,\ \rho_{ij},\ \rho_i^u > 0$ and functions $\beta_i \in \mathcal{KL}$ and $\gamma_{ij}^d, \gamma_i^u \in \mathcal{K}$, $i,j = 1,\ldots,n$, $i \ne j$ such that for initial functions $\|\xi_i\|_{[-\theta,0]} \le \rho_i$, for inputs $\|x_j\|_{[-\theta,\infty)} \le \rho_{ij}$, $\|u_i\|_\infty \le \rho_i^u$ and for all $t \in \mathbb{R}_+$ it holds

$$\begin{aligned}|x_i(t)| \le \max\Big\{&\beta_i\left(\|\xi_i\|_{[-\theta,0]}, t\right), \max_{j\ne i} \gamma_{ij}^d\left(\|x_j\|_{[-\theta,\infty)}\right), \\ &\gamma_i^u\left(\|u\|_\infty\right)\Big\},\end{aligned} \tag{11}$$

where $\|x_j\|_{[-\theta,\infty)} := \sup_{t\in[-\theta,\infty)} |x_j(t)|$.

As in the delay-free case, Lyapunov functions are a useful tool to investigate stability of systems with time-delays, where one can use Lyapunov–Razumikhin functions or Lyapunov–Krasovskii functionals (see [17, 27]). In this paper, we only use Lyapunov–Razumikhin functions for the stability analysis. An approach by the help of Lyapunov–Krasovskii functionals can be found in [17] and [9]. The existence of an ISS Lyapunov–Razumikhin function implies ISS for systems of the form (7). This was shown in [27] and can be transferred to LISS in a similar way. For the definition of LISS Lyapunov–Razumikhin functions, we introduce the upper right-hand side derivative of a locally Lipschitz continuous function $V : \mathbb{R}^N \to \mathbb{R}_+$ along the solution $x(t)$, which is defined by

$$D^+V(x(t)) = \limsup_{h\to 0^+} \frac{V(x(t+h)) - V(x(t))}{h}.$$

For interconnected time-delay systems, the LISS Lyapunov–Razumikhin functions are defined in the following way:

Definition 7 We assume that for each subsystem of the interconnected system (10) there exists a function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous and positive definite. Then, for $i = 1,\ldots,n$ the function V_i is called an *LISS Lyapunov-Razumikhin function* for the i-th subsystem of (10) if there exist constants $\rho_i,\ \rho_{ij},\ \rho_i^u > 0$ and functions $\beta_i \in \mathcal{KL},\ \gamma_{ij}^d, \gamma_i^u \in \mathcal{K} \cup \{0\},\ \mu_i \in \mathcal{K}$, $i,j = 1,\ldots,n$ such that

$$\psi_{1i}(|x_i|) \le V_i(x_i) \le \psi_{2i}(|x_i|),\ \forall x_i \in \mathbb{R}^{N_i}, \tag{12}$$

$$V_i(x_i) \ge \max\left\{\max_j \gamma_{ij}^d\left(||V_j^d(x_j)||\right), \gamma_i^u(|u|)\right\} \\ \Rightarrow\ D^+V_i(x_i) \le -\mu_i(V_i(x_i)) \tag{13}$$

for all initial functions $\|\xi_i\|_{[-\theta,0]} \le \rho_i$, for all inputs $|x_j| \le \rho_{ij},\ |u_i| \le \rho_i^u$ and for all $t \in \mathbb{R}_+$, where $V_j^d(x_j(t)) := V_j(x_j(t+\tau)),\ \tau \in [-\theta, 0]$ and $||V_j^d(x_j)|| := \max_{t-\theta \le s \le t} |V_j(x_j(s))|$.

Furthermore, we define the gain-matrix for time-delay systems by $\overline{\Gamma} := (\gamma_{ij}^d)_{n\times n}$ and the map $\overline{\Gamma} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\overline{\Gamma}(s) := \left(\max_j \gamma_{1j}^d(s_j), \ldots, \max_j \gamma_{nj}^d(s_j)\right)^T,\ s \in \mathbb{R}_+^n.$$

With help of the following theorem, we can check whether an interconnected system with time-delays is LISS.

Theorem 2 *Consider the interconnected system (10), where each subsystem has a LISS Lyapunov-Razumikhin function V_i. If the corresponding gain-operator $\overline{\Gamma}$ satisfies the local small-gain condition from Definition 1, then there exist constants $\rho,\ \rho_u > 0$ such that the whole system of the form (7) is LISS.*

This follows from Theorem 1 in [9] with the corresponding changes according to the LISS property.

Theorems 1 and 2 will be used in the following section for a stability analysis of production networks.

3 Modeling and stability analysis of production networks

In this section, we model general production networks and perform a stability analysis, where the methods and tools presented in the previous section are used. We will derive a sufficient condition, which guarantees stability of a general network.

3.1 Description and modeling of a general production network

We consider a production network, consisting of n market entities, which can be raw material suppliers (e.g., extracting or agricultural companies), producers, distributors, and consumers, for example. Each entity is understood as a subsystem of the whole network. For simplicity, we assume that there is only one unified type of material, i.e., all primary products, used in the production network, can be measured as a number of units of this unified material.

The state of the i-th subsystem at time $t \in \mathbb{R}_+$ is the quantity of unprocessed material within the i-th subsystem at time t. It will be denoted by $x_i(t)$. The state of the whole network at time t is denoted by $x(t) = (x_1(t), \ldots, x_n(t))^T$. A subsystem can get material from an external source, which is denoted by u_i, and from subsystems of the network (internal inputs).

3.1.1 Modeling without time-delays

At first, we consider a production network without transportation times and use ordinary differential equations to model it. Let the i-th subsystem processes the raw material from its inventory with rate $\tilde{f}_{ii}(t, x(t)) \ge 0$ and sends the produced goods (measured in units of unified material) to the j-th subsystem with rate $\tilde{f}_{ji}(t, x(t))$. Thus, the total rate of the distribution from the i-th subsystem to other subsystems is $\sum_{j=1}^n \tilde{f}_{ji}(t, x(t))$ and the rest is sent to some customers not considered in the network.

For general functions $\tilde{f}_{ji}$ it is hard to derive stability conditions. Therefore, we will investigate the special case $\tilde{f}_{ji}(t, x(t)) = c_{ji}(x(t))\tilde{f}_i(x_i(t)),\ c_{ji}(x) \in \mathbb{R}_+$ and $\tilde{f}_{ii}(t, x(t)) = \tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t)),\ \tilde{c}_{ii}(x) \in \mathbb{R}_+$, where $\tilde{f}_i(x_i(t)) \in \mathcal{K}$ is proportional to the processing rate of the system, $c_{ji}(x(t))$, $i \neq j$ are some positive distribution coefficients and $\tilde{c}_{ii}(x(t)) \ge 0$. We interpret the constant distribution coefficients as central planning, and on the other hand, variable distribution coefficients can be used for some autonomous control method, e.g., the QLE.

Under these assumptions, the dynamics of the i-th subsystem is described by ordinary differential equations as in (2):

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t)) + u_i(t) - \tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t)), \quad i = 1, \ldots, n. \tag{14}$$

Denoting $c_{ii} := -\tilde{c}_{ii}$ we rewrite the above equations in a vector form as an interconnected system of the form (1)

$$\dot{x}(t) = C(x(t))\tilde{f}(x(t)) + u(t), \tag{15}$$

where $\tilde{f}(x(t)) = (\tilde{f}_1(x_1(t)), \ldots, \tilde{f}_n(x_n(t)))^T$, $u(t) = (u_1(t), \ldots, u_n(t))^T$ and $C(x) \in \mathbb{R}^{n\times n}$.

This model will be used in the next subsection for a stability analysis of general production networks.

3.1.2 Modeling with time-delays

Now, we model general production networks with transportation times using time-delay systems. The time needed for the transportation of material from the j-th to the i-th entity is denoted by $\tau_{ij} \in \mathbb{R}_+$. Then, the dynamics of the i-th subsystem can be described by retarded differential equations similar to (14):

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t-\tau_{ij})) + u_i(t) - \tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t)), \ i = 1, \ldots, n. \tag{16}$$

Here, the external input and the processing rate do not depend on any time-delay, but the internal inputs from other subsystems do, represented by the terms $c_{ij}(x(t))\tilde{f}_j(x_j(t-\tau_{ij}))$. This means that the input of subsystem i at time t from subsystem j is the amount of material that was sent by the j-th subsystem at the time $t - \tau_{ij}$. The terms $c_{ij}(x(t))$ can also depend on $x_j(t - \tau_{ij})$, but we write $c_{ij}(x(t))$ for short.

In the next subsection, we perform a stability analysis for such systems, where we use the Lyapunov–Razumikhin approach.

3.2 Stability analysis

For the stability analysis, we apply the framework shown in Fig. 1. This framework is based on the result of Theorem 2. At first, we model the production network using ODEs. Then, we look for the ISS-Lyapunov function for each subsystem described in (14). To this end, we iteratively choose some candidate to be ISS-Lyapunov function and the corresponding Lyapunov gains and check whether the conditions on an ISS Lyapunov function are satisfied. If an ISS-Lyapunov function is found, then we verify the small-gain condition. If it is satisfied, then Theorem 2 is applied to establish ISS.

3.2.1 Stability analysis of production networks modeled without time-delays

At first, we consider the case $\tilde{f}_i \in \mathcal{K}_\infty$, $i = 1, \ldots, n$, in particular $\tilde{f}_i$ are unbounded. Later, we will show how the same method can be applied with minimal modifications for bounded $\tilde{f}_i \in \mathcal{K} \setminus \mathcal{K}_\infty$.

Note that the conditions $\tilde{f}_i \in \mathcal{K}_\infty$, for all $x > 0$ $c_{ii}(x) < 0$ and $c_{ij}(x) \geq 0$, $i \neq j$ imply that if $x(0) \geq 0$ (that is $x_i(0) \geq 0$ for all $i = 1, \ldots, n$) and $u(t) \geq 0$, for all $t > 0$, then $x(t) \geq 0$ for all $t > 0$.

Thus, $\mathbb{R}^n_+ = [0, \infty)^n$ is invariant under the flow of internal and external inputs (if the external inputs are positive). One can perform the following analysis and prove the results in $\mathbb{R}^n$. Since we are interested in the stability analysis of production networks, it is enough to perform the analysis in this paper in $\mathbb{R}^n_+$.

There are many candidates to be ISS-Lyapunov functions. We consider the one that is easy to check, i.e., $V_i(x_i) = |x_i| = x_i$ for the i-th entity. Obviously, $V_i(x_i)$ satisfies the condition (3). To prove that the condition (4) holds, we choose $\forall s \in \mathbb{R}_+$ the functions γ_{ij}, γ_i, μ_i (see Definition 3) as

$$\gamma_{ij}(s) := \tilde{f}_i^{-1}\left(\frac{a_i}{a_j}\frac{1}{1+\delta_j}\tilde{f}_j(s)\right), \quad \gamma_i(s) := \tilde{f}_i^{-1}\left(\frac{1}{r_i}s\right), \tag{17}$$

where δ_j, a_j, $j = 1, \ldots, n$ and r_i are positive reals. It follows from (17) that

$$x_i \geq \gamma_{ij}(x_j) \ \Rightarrow \tilde{f}_j(x_j) \leq \frac{a_j}{a_i}(1+\delta_j)\tilde{f}_i(x_i),$$
$$x_i \geq \gamma_i(|u_i|) \ \Rightarrow |u_i| \leq r_i\tilde{f}_i(x_i).$$

Using the inequalities from the right-hand side of the implications above and assuming that the following condition holds $\forall s \in \mathbb{R}^n_+$, for some $h_i > 0$,

$$\sum_{j=1, j\neq i}^{n} c_{ij}(x)\frac{a_j}{a_i}(1+\delta_j) + c_{ii}(x) + r_i \leq -h_i, \tag{18}$$

we obtain that for all $x_i \in \mathbb{R}_+ : V_i(x_i) \geq \max\{\max_{j\neq i}\gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\}$ (compare with (4)) it holds

$$\frac{dV_i(x_i(t))}{dt} = \sum_{j=1}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t)) + u_i(t) \leq \left(\sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\frac{a_j}{a_i}(1+\delta_j) + c_{ii}(x(t)) + r_i\right)\tilde{f}_i(x_i(t)) \leq -\mu_i(V_i(x_i(t))),$$

where $\mu_i(r) := h_i\tilde{f}_i(r), r \in \mathbb{R}_+$ and thereby condition (4) is satisfied. Thus, under the condition (18), $V_i(x_i) = |x_i|$ is an

ISS Lyapunov function for the i-th entity with the gains, given by (17).

To check whether the interconnected system (15) is ISS, we need to verify the small-gain condition. It is well known that this condition is equivalent to the cycle condition (see [10]): for all $p = 2, \ldots, n$, for all $(k_1, \ldots, k_p) \in \{1, \ldots, n\}^p$, where $k_1 = k_p$, it holds $\forall s \in \mathbb{R}_+ \backslash \{0\}$

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \ldots \circ \gamma_{k_{p-1}k_p}(s) < s. \tag{19}$$

Consider a composition $\gamma_{k_1k_2} \circ \gamma_{k_2k_3}$, then it holds

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3}(s) = \tilde{f}_{k_1}^{-1}\left(\frac{a_{k_1}}{a_{k_2}}\frac{1}{1+\delta_{k_2}}\tilde{f}_{k_2}\left(\tilde{f}_{k_2}^{-1}\left(\frac{a_{k_2}}{a_{k_3}}\frac{1}{1+\delta_{k_3}}\tilde{f}_{k_3}(s)\right)\right)\right)$$
$$= \tilde{f}_{k_1}^{-1}\left(\frac{a_{k_1}}{a_{k_3}}\frac{1}{(1+\delta_{k_3})(1+\delta_{k_2})}\tilde{f}_{k_3}(s)\right).$$

In the same way, we obtain the expression for the cycle condition in (19) (here we use that $k_1 = k_p$):

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \ldots \circ \gamma_{k_{p-1}k_p}(s)$$
$$= \tilde{f}_{k_1}^{-1}\left(\frac{1}{\prod_{i=2}^{p}(1+\delta_{k_i})}\tilde{f}_{k_1}(s)\right) < s.$$

Thus, the small-gain condition (19) holds true for all $\delta_i > 0$, and by Theorem 1, the whole system is ISS.

If we assume that the c_{ij} are bounded, i.e., there exists $M > 0$ such that for all $s \in \mathbb{R}_+^n$: $c_{ij}(s) \le M$ for all $i,j = 1, \ldots, n$, $i \ne j$, then the inequality (19) can be simplified:

$$\forall w_i > 0\ \exists \delta_j > 0, j = 1, \ldots, n:$$
$$\sum_{j=1, j\ne i}^{n} c_{ij}(s)\frac{a_j}{a_i}\delta_j \le M\left(\sum_{j=1, j\ne i}^{n}\frac{a_j}{a_i}\delta_j\right) < w_i.$$

Using these estimates, we can rewrite (18) as

$$\sum_{j=1, j\ne i}^{n} c_{ij}(s)a_j \le -c_{ii}(s)a_i - \varepsilon_i,$$

where $\varepsilon_i = a_i(r_i + h_i + w_i)$. In matrix notation, with $a = (a_1, \ldots, a_n)^T$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$, it takes the form

$$C(s)a < -\varepsilon. \tag{20}$$

We summarize our investigations in the following proposition.

Proposition 1 *Consider a network as in (14) and assume that the c_{ij} are bounded for all $i,j = 1, \ldots, n$, $i \ne j$. If there exist $a \in \mathbb{R}^n$, $\varepsilon \in \mathbb{R}^n$, $a_i > 0$, $\varepsilon_i > 0$, $i = 1, \ldots, n$ such that the condition $C(s)a < -\varepsilon$ holds for all $t > 0$* and *$s \in \mathbb{R}_+^n$, then the whole network (15) is ISS.*

Remark 2 If the matrix C does not depend on s, then the condition $Ca < -\varepsilon$ is equivalent to $Ca < 0$ (with a, ε as in the proposition above). But if $C = C(s)$, then the existence of a positive vector a, $Ca < 0$ is not enough to guarantee ISS of the system (15).

Remark 3 Recall that for the case, when C is a constant matrix with negative elements on the main diagonal and all other elements are nonnegative, C is diagonal dominant (see, e.g., [3]), if it holds $c_{ii} + \sum_{j \ne i} c_{ij} < 0$ for all $i = 1, \ldots, n$. In this case, one can easily prove with help of Gershgorin circle theorem (see [3], Fact 4.10.17), that C is Hurwitz. Similarly, the previous condition can be replaced with another one: there are n numbers $a_i > 0$ such that $c_{ii}\ a_i + \sum_{j \ne i} c_{ij}a_j < 0$ for all $i = 1, \ldots, n$ (which is equivalent to the existence of a positive vector a such that $Ca < 0$). In this case the matrix is also Hurwitz (see, e.g., [13]).

Now, we consider $\tilde{f}_i \in \mathcal{K} \setminus \mathcal{K}_\infty$, $i = 1, \ldots, n$, i.e., functions $\tilde{f}_i$ are monotonously increasing, but only up to a certain limit $\alpha_i := \sup_{x_i}\{\tilde{f}_i(x_i)\}$. For such $\tilde{f}_i$ the global ISS property cannot be achieved, but we can establish the LISS property. We choose again the function $V_i = |x_i| = x_i$ as LISS Lyapunov function candidate for the i-th subsystem and the corresponding gains $\forall s \in \mathbb{R}_+$ as follows

$$\gamma_{ij}(s) := \tilde{f}_i^{-1}\left(\frac{\alpha_i}{\alpha_j}\frac{1}{1+\delta}\tilde{f}_j(s)\right),\ \gamma_i(s) := \tilde{f}_i^{-1}\left(\frac{\alpha_i}{\|u_i\|_\infty r_i}s\right).$$

Note that in contrast to the previous case, where the coefficients a_i involved in the gain functions were chosen arbitrarily, the α_j are taken from the boundedness assumptions on the functions $\tilde{f}_i$. The reason is to obtain a range of a function $\frac{\alpha_i}{\alpha_j}\tilde{f}_j(s)$ equal to the domain of definition of $\tilde{f}_i^{-1}$.

Applying the same methods as for $\tilde{f}_i \in \mathcal{K}_\infty$, we obtain the following proposition:

Proposition 2 *Consider a network as in (14). Let $\tilde{f}_j \in \mathcal{K} \setminus \mathcal{K}_\infty$, and $\alpha_j := \sup_{x_j \in \mathbb{R}}\{\tilde{f}_j(x_j)\}$, $j = 1, \ldots n$, $\alpha := (\alpha_1, \ldots, \alpha_n)^T$. If there exist $g \in \mathbb{R}_+^n$, $g_i > 0$ and $\varepsilon \in \mathbb{R}^n$, $\varepsilon_i > 0$, $i = 1, \ldots, n$ such that*

$$C(s)\alpha + g < -\varepsilon, \forall s \in \mathbb{R}_+^n, \tag{21}$$

then the whole network (15) is LISS. Furthermore, the constants ρ and ρ_u from the Definition 1 can be chosen as $\rho := \infty$, $\rho_u := \min_{i=1,\ldots,n} g_i$ and (21) holds for all $u \in L_\infty(\mathbb{R}_+, \mathbb{R}_+^n)$: $\|u_i\|_\infty \le g_i$, for all $i = 1, \ldots, n$.

Remark 4 The stability analysis for functions $\tilde{f}_i \in \mathcal{K}$ is skipped here, because some more technical details are necessary that would increase the size of the paper drastically. The result is similar to Proposition 2.

3.2.2 Stability analysis of production networks with time-delays

Now, we perform a stability analysis for general production networks with transportation times modeled by time-delay systems of the form (16), where we use the tools presented in the Sect. 2.2.

Consider the case $\tilde{f}_i \in \mathcal{K}_\infty,\ i = 1, \ldots, n$, in particular $\tilde{f}_i$ are unbounded. We choose $V_i(x_i) = |x_i| = x_i$ as an ISS-Lyapunov–Razumikhin function candidate for the i-th entity. Obviously, $V_i(x_i)$ satisfies the condition (12). To prove that the condition (13) holds, we choose the functions γ^d_{ij} and γ^u_i as γ_{ij}, γ_i in (17), where $\gamma^d_{ii} \equiv 0$ because there is no time-delay in the internal dynamics (see the term $\tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t))$ in the model). The difference to (17) is that the time-delay is taken into account in the argument of the gains. From the condition (13), we have

$$V_i(x_i) \geq \max\left\{\max_j \gamma^d_{ij}(\|V^d_j(x_j)\|), \gamma^u_i(|u|)\right\},$$

where $V^d_j(x_j(t)) = V_j(x_j(t - \tau_{ij}))$ and $\|V^d_j(x_j)\| = \max_{t-\tau_{ij} \leq s \leq t}\left|V_j(x_j(s))\right|$. This means $\gamma^d_{ij}(\|V^d_j(x_j)\|) \geq \gamma_{ij}(V_j(x_j))$ and furthermore for $\tau_{ij} > \tilde{\tau}_{ij} \Rightarrow \gamma^d_{ij}(|V_j(x_j(t-\tau_{ij}))|) \geq \gamma_{ij}(|V_j(x_j(t - \tilde{\tau}_{ij}))|)$.

From the definition of the gains, we get by application of the Theorem 2 the following proposition by similar calculations as for the stability analysis based on ODEs.

Proposition 3 *Consider a network as in (16).*

1. *Assume that the c_{ij} are bounded for all $i, j = 1, \ldots, n,\ i \neq j$. If there exist $a \in \mathbb{R}^n,\ \varepsilon \in \mathbb{R}^n,\ a_i > 0,\ \varepsilon_i > 0,\ i = 1, \ldots, n$ such that the condition $C(s)a < -\varepsilon$ holds $\forall t > 0, \forall s \in \mathbb{R}^n_+$, then the whole network is ISS.*
2. *Let $\tilde{f}_j \in \mathcal{K} \setminus \mathcal{K}_\infty$, and $\alpha_j := \sup_{x_j}\{\tilde{f}_j(x_j)\},\ j = 1, \ldots n$, $\alpha := (\alpha_1, \ldots, \alpha_n)^T$. If there exist $g \in \mathbb{R}^n_+,\ g_i > 0$ and $\varepsilon \in \mathbb{R}^n,\ \varepsilon_i > 0,\ i = 1, \ldots, n$ such that*

$$C(s)\alpha + g < -\varepsilon, \forall s \in \mathbb{R}^n_+, \tag{22}$$

then the whole network is LISS. Furthermore, the constants ρ and ρ_u from the Definition 2 can be chosen as $\rho := \infty$ and $\rho_u := \min_{i=1,\ldots,n} g_i$ and (22) holds for all $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^n_+) :\ \|u_i\|_\infty \leq g_i$, for all $i = 1, \ldots, n$.

These results are applied to a certain scenario of a production network in the following section.

4 Example of a certain scenario of a production network

4.1 System without time-delays

We consider a certain scenario of a production network without transportation times as in Fig. 2. There, the numbers of the nodes are given in the centers of the corresponding circles. The first entity gets some raw material from an external supplier, denoted by u. At each entity the material will be processed with the rates $c_{ii}\tilde{f}_i = c_{ii}q_i\tilde{f},\ q_i \geq 0$ and immediately sent to the entities according to the network topology in Fig. 2 with certain distribution coefficients c_{ij}. One half of the production of the entity four will be sent to customers, not considered in the network. The distribution coefficients are given by

$$C(x(t)) = \begin{pmatrix} -2 & 0 & 0 & 0.5 \\ c_{21}(x(t)) & -1.5 & 0 & 0 \\ c_{31}(x(t)) & 0 & -2 & 0 \\ 0 & 1 & 1 & -2.5 \end{pmatrix}, \tag{23}$$

where we implement the queue length method by choosing

$$c_{21}(x(t)) := \frac{A}{A+B}, \quad c_{31}(x(t)) := \frac{B}{A+B},$$
$$\text{where } A := \frac{c_{22}q_2}{x_2(t)+\varepsilon},\ B := \frac{c_{33}q_3}{x_3(t)+\varepsilon}.$$

The term $\varepsilon > 0$ assures that the $c_{ij}(x(t))$ are well defined and for simplicity one can choose ε close to zero. Note that $c_{21} + c_{31} \equiv 1$.

To analyze whether the network has the ISS property, we only have to check the condition (20), which can be easily verified with $a_i = 1,\ i = 1, \ldots, 4$. By Proposition 2, the whole network is ISS. The gains are of the form

$$\gamma_{ij}(s) := \tilde{f}_i^{-1}\left(\frac{1}{1+\delta_j}\tilde{f}_j(s)\right) = \tilde{f}^{-1}\left(\frac{q_i}{q_j}\frac{1}{1+\delta_j}\tilde{f}(s)\right), s \in \mathbb{R}_+,$$

where $\delta_j > 0$. For example, let $\tilde{f}(s) = \sqrt{s}$ and $q_i = 1, i = 1, \ldots, 4$. Then, we have

$$\gamma_{ij}(s) = \frac{1}{(1+\delta_j)^2}s,\ s \in \mathbb{R}_+.$$

Such a function $\tilde{f}$ describes a typical production policy.

The differential equations that describe the systems behavior are of the form

$$\dot{x}_1(t) = u(t) + \frac{1}{2}\sqrt{x_4(t)} - 2\sqrt{x_1(t)},$$

$$\dot{x}_2(t) = \frac{\frac{1.5}{x_2(t)}}{\frac{1.5}{x_2(t)} + \frac{2}{x_3(t)}}\sqrt{x_1(t)} - 1.5\sqrt{x_2(t)},$$

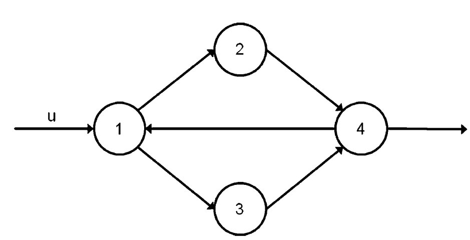


Fig. 2 Example of a scenario of a production network

$$\dot{x}_3(t) = \frac{\frac{2}{x_3(t)}}{\frac{1.5}{x_2(t)} + \frac{2}{x_3(t)}} \sqrt{x_1(t)} - 2\sqrt{x_3(t)},$$
$$\dot{x}_4(t) = \sqrt{x_2(t)} + \sqrt{x_3(t)} - 2.5\sqrt{x_4(t)}.$$

Let the initial state be given by $x(0) = (2, 5, 4, 3)^T$ and the input function be $u = 10 \cdot (\sin(t) + 1)$ that is a fluctuation of customer demand from 0 to 20. Note that one can choose any other initial state and input u for which the condition (21) is satisfied. Then, we get the stable behavior, displayed in Figs. 3 and 4, where a simulation is performed with Matlab.

If the distribution coefficients are chosen as $c_{11} = -1$, $c_{22} = -1$, $c_{33} = -1$, $c_{44} = -1$, i.e., the condition (20) is not satisfied, then we cannot make a statement about stability. Indeed, in this case, we get the following unstable behavior displayed in Figs. 5 and 6. It means that the number of unprocessed parts within a subsystem increases up to infinity.

4.2 System with time-delays

Now, we consider the same scenario of a production network as in Fig. 2, but with transportation times. The distribution coefficients c_{ij} for the stable situation are given by (23) with c_{21} and c_{31} which represent the queue length method and take into account time-delays:

$$c_{21}(x(t)) := \frac{\widetilde{A}}{\widetilde{A} + \widetilde{B}}, \quad \text{where } \widetilde{A} := \frac{c_{22}q_2}{x_2(t - \tau_{21}) + \varepsilon},$$
$$\widetilde{B} := \frac{c_{33}q_3}{x_3(t - \tau_{21}) + \varepsilon},$$
$$c_{31}(x(t)) := \frac{\overline{B}}{\overline{A} + \overline{B}}, \quad \text{where } \overline{A} := \frac{c_{22}q_2}{x_2(t - \tau_{31}) + \varepsilon},$$
$$\overline{B} := \frac{c_{33}q_3}{x_3(t - \tau_{31}) + \varepsilon}.$$

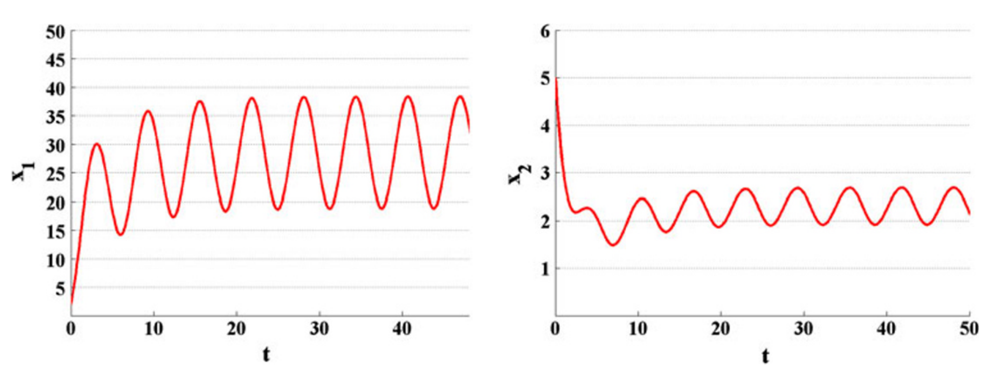
Now, we choose $\tilde{f}_i(s) = q_i\sqrt{s}$ with $q_1 = 3$, $q_2 = 2$, $q_3 = 1.5$, $q_4 = 1.6$. The condition (20) is satisfied, which can be easily checked, and therefore, the network has the ISS property. The retarded differential equations of the system are of the form

$$\dot{x}_1(t) = u(t) + \frac{1.6}{2}\sqrt{x_4(t - \tau_{14})} - 6\sqrt{x_1(t)},$$
$$\dot{x}_2(t) = \frac{\frac{3}{x_2(t-\tau_{21})}}{\frac{3}{x_2(t-\tau_{21})} + \frac{3}{x_3(t-\tau_{21})}} 3\sqrt{x_1(t - \tau_{21})} - 3\sqrt{x_2(t)},$$
$$\dot{x}_3(t) = \frac{\frac{3}{x_3(t-\tau_{31})}}{\frac{3}{x_2(t-\tau_{31})} + \frac{3}{x_3(t-\tau_{31})}} 3\sqrt{x_1(t - \tau_{31})} - 3\sqrt{x_3(t)},$$
$$\dot{x}_4(t) = 2\sqrt{x_2(t - \tau_{42})} + 1.5\sqrt{x_3(t - \tau_{43})} - 4\sqrt{x_4(t)}.$$

We choose $\tau_{ij} = 2$ and the initial function $x(s) \equiv (2, 5, 4, 3)^T$, $s \in [-2, 0]$. The input function is given by the constant function $u \equiv 20$ in contrast to the oscillating input used before. Then, we get the stable behavior, displayed in Fig. 7. Although we choose a constant input, we observe an oscillating behavior of the number of unprocessed parts of the subsystems. The reason is the implemented queue length method in the terms $c_{21}(x(t))$ and $c_{31}(x(t))$: Here, only the number of unprocessed parts at the time $t - \tau_{21}$ or $t - \tau_{31}$ is used for the calculation of the distribution coefficients $c_{i1}(x(t))$, $i = 2, 3$. The number of unprocessed parts, which has been sent during the time $(t - \tau_{i1}$,0] and has not yet been arrived at subsystem two or three, is not taken into account. Then, it happens that more parts are sent to a subsystem with larger queue than to the other subsystem until the distribution coefficients of both subsystems, depending on the number of unprocessed parts at time $t - \tau_{i1}$, are equal. After this point, the proportionally higher number of sent parts arrive at the subsystem, which increases continuously the queue length and leads to a smaller distribution coefficient c_{i1} in contrast to the distribution coefficient of the other subsystem. Now, the procedure goes on in the opposite direction until the distribution coefficients are equal again. This cycle repeats and causes the observed oscillating behavior.

Now, we increase the time-delays by choosing $\tau_{ij} = 4$ and the initial function $x(s) \equiv (2, 5, 4, 3)^T$, $s \in [-4, 0]$. Furthermore, we choose $\varepsilon = 0.001$ to assure that the distribution coefficients c_{1i} are well defined. All other

Fig. 3 Stable evolution of the amount of unprocessed parts within subsystems one and two



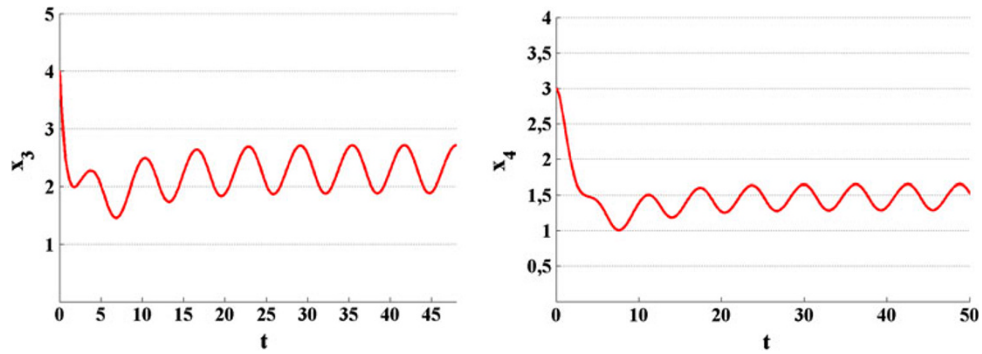


Fig. 4 Stable evolution of the amount of unprocessed parts within subsystems three and four

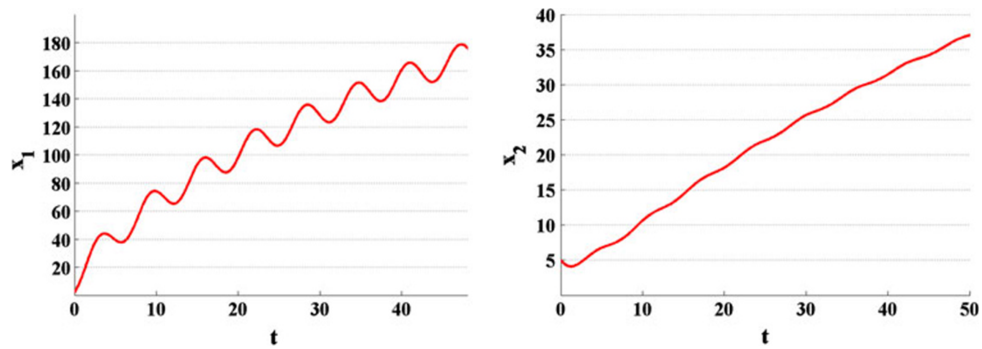


Fig. 5 Unstable evolution of the amount of unprocessed parts within subsystems one and two

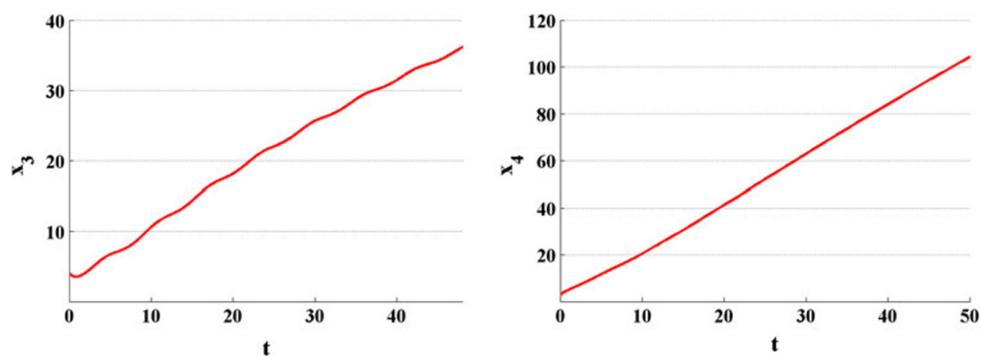


Fig. 6 Unstable evolution of the amount of unprocessed parts within subsystems three and four

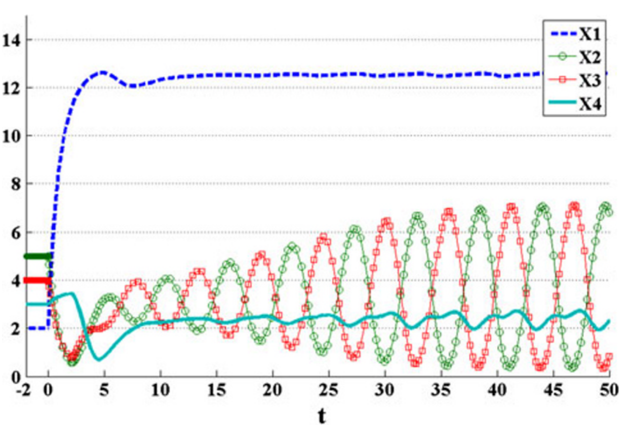


Fig. 7 Stable evolution with the time-delays $\tau_{ij} = 2$

parameters are the same. Then, we get the behavior of the number of unprocessed parts of the subsystems displayed in Fig. 8. The increased time-delays $\tau_{ij} = 4$ cause higher amplitudes, i.e., larger maximal values of the number of unprocessed parts of a subsystem in contrast to the time-delays $\tau_{ij} = 2$ used in Fig. 7. Furthermore, as a result of this increased oscillations, we observe that for some time intervals, the number of unprocessed parts of subsystem two and three equals or is close to zero, which means that the entities do not produce parts in these time intervals. In the conclusions, we provide some ideas to avoid such negative outcomes.

5 Summary

5.1 Conclusions

We have modeled and investigated general production networks in view of stability with and without transportation times. Two modeling methods were presented:

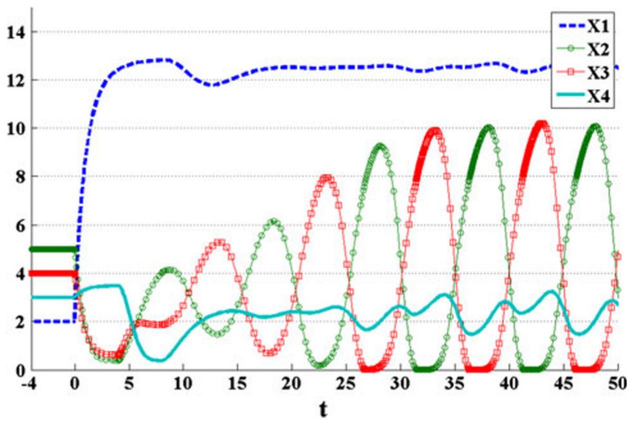


Fig. 8 Stable evolution with the time-delays $\tau_{ij} = 4$

modeling by differential equations with and without time-delays. They were used to model general production networks, where an autonomous control method, the queue length method, was implemented. Based on these models, we have presented tools to perform a stability analysis using (L)ISS-Lyapunov or (L)ISS-Lyapunov-Razumikhin functions. We have derived a condition that guarantees that a network possesses the (L)ISS property. This result was applied to a scenario of a production network with and without transportation times. Here, we have found out that the maximum number of unprocessed parts of a subsystem with time-delays can be higher than that of a subsystem without time-delays. Furthermore, we have observed an oscillating behavior of the number of unprocessed parts of a subsystem with time-delays, which was caused by the modeled queue length method. The larger the time-delay, the higher is this oscillating behavior and could cause downtimes of the production.

5.2 Future work

The choices of the parameters c_{ij} for the modeling of the queue length method can be changed: the number of parts which are on the way to a subsystem, but not yet arrive there, can be taken into account. This means that full information access of the market entities of a network is necessary, which is not always available. This problem should be investigated. Another way of modeling the queue length method can be performed by using switched systems [16]. For such modeling method, the tools to perform a stability analysis for general networks have to be developed. One can extend the modeling of production networks by taking into account state jumps, e.g., loading and unloading processes, one can use hybrid or impulsive systems with and without time-delays [8]. Then, the developed dwell-time condition plays a significant role and should be investigated in more detail.

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